

SYMPLECTIC BRANCHING LAWS AND HERMITIAN SYMMETRIC SPACES

BENJAMIN SCHWARZ AND HENRIK SEPPÄNEN*

ABSTRACT. Let G be a complex simple Lie group, and let $U \subseteq G$ be a maximal compact subgroup. Assume that G admits a homogeneous space $X = G/Q = U/K$ which is a compact Hermitian symmetric space. Let $\mathcal{L} \rightarrow X$ be the ample line bundle which generates the Picard group of X . In this paper we study the restrictions to K of the family $(H^0(X, \mathcal{L}^k))_{k \in \mathbb{N}}$ of irreducible G -representations. We describe explicitly the moment polytopes for the moment maps $X \rightarrow \mathfrak{k}^*$ associated to positive integer multiples of the Kostant-Kirillov symplectic form on X , and we use these, together with an explicit characterization of the closed $K^{\mathbb{C}}$ -orbits on X , to find the decompositions of the spaces $H^0(X, \mathcal{L}^k)$. We also construct a natural Okounkov body for \mathcal{L} and the K -action, and identify it with the smallest of the moment polytopes above. In particular, the Okounkov body is a convex polytope. In fact, we even prove the stronger property that the semigroup defining the Okounkov body is finitely generated.

1. INTRODUCTION

In this paper we consider the following setting. Let G be a complex simple Lie group, and assume that G admits a quotient $X := G/Q$ which is a compact Hermitian symmetric space. Then Q is a maximal parabolic subgroup. Moreover, we can write X as $X = U/K$, where $U \subseteq G$ is a maximal compact subgroup, and $K := U \cap Q$. The Picard group of X , which is isomorphic to the group of holomorphic characters $Q \rightarrow \mathbb{C}^\times$, is \mathbb{Z} . Let $\mathcal{L} \rightarrow X$ be the ample generator for the Picard group. We are concerned with the decomposition under K of the irreducible G -representations given by the family $H^0(X, \mathcal{L}^k)$, where $k \in \mathbb{N}$.

In order to put our approach to the decomposition problem into its proper framework we make a small digression into a more general setting. For a more thorough treatment we refer to [S95] and the references therein. We now temporarily let K denote an arbitrary compact Lie group, and assume that K acts holomorphically and in a Hamiltonian fashion on the connected compact Kähler manifold (M, ω) with moment map $\tau \rightarrow \mathfrak{k}^*$. Assume that (M, ω) admits a prequantum line

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bundle \mathcal{L} . The action of K then lifts to an action on \mathcal{L} , and hence we have a representation of K on the space $H^0(M, \mathcal{L})$. One may then ask how $H^0(M, \mathcal{L})$ decomposes under K . For this purpose it is useful to realize the irreducible K -representation of highest weight ξ as the space of holomorphic sections $H^0(\mathcal{O}_\xi^K, \mathcal{L}_\xi)$, where $\mathcal{O}_\xi^K \subseteq \mathfrak{k}^*$ is the coadjoint orbit through ξ , and \mathcal{L}_ξ is the prequantum line bundle attached to the Kostant-Kirillov symplectic form on \mathcal{O}_ξ^K . It is well-known that the multiplicity of the representation $H^0(\mathcal{O}_\xi^K, \mathcal{L}_\xi)$ in $H^0(M, \mathcal{L})$ is given by the dimension of the space, $H^0(M \times \overline{\mathcal{O}_\xi^K}, \mathcal{L} \boxtimes \overline{\mathcal{L}_\xi^*})^K$, of K -invariant holomorphic sections of the line bundle $\mathcal{L} \boxtimes \overline{\mathcal{L}_\xi^*} \rightarrow M \times \overline{\mathcal{O}_\xi^K}$. Here $\overline{\mathcal{O}_\xi^K}$ denotes the topological space \mathcal{O}_ξ^K equipped with the reverse complex structure and the symplectic form given by -1 times the Kostant-Kirillov form. The fibre over $x \in \mathcal{O}_\xi^K$ of the line bundle $\overline{\mathcal{L}_\xi^*}$ consists of the space of antilinear complex-valued functionals on $(\mathcal{L}_\xi)_x$ (cf. [GS82]). The group K now acts holomorphically and in a Hamiltonian fashion on $M \times \overline{\mathcal{O}_\xi^K}$ with moment map $\tau^\xi : M \times \overline{\mathcal{O}_\xi^K} \rightarrow \mathfrak{k}^*$ given by $\tau^\xi(m, f) := \tau(m) - f$.

An obvious question is whether the space $H^0(M \times \overline{\mathcal{O}_\xi^K}, \mathcal{L} \boxtimes \overline{\mathcal{L}_\xi^*})^K$ can be interpreted as the space of all holomorphic sections of some line bundle over some “quotient” of $M \times \overline{\mathcal{O}_\xi^K}$ by K . Indeed, this holds for the Mumford quotient

$$(M \times \overline{\mathcal{O}_\xi^K})_0 := (M \times \overline{\mathcal{O}_\xi^K})_{ss} // K^\mathbb{C},$$

where $(M \times \overline{\mathcal{O}_\xi^K})_{ss}$ is the open subset consisting of the *semistable points* of $(M \times \overline{\mathcal{O}_\xi^K})$. An interesting feature, and one which links the Mumford quotient to the symplectic geometry, is that $(M \times \overline{\mathcal{O}_\xi^K})_0$ is homeomorphic to the topological quotient $(\tau^\xi)^{-1}(0)$ (cf. [S95, Thm. 2.5]), the *symplectic reduction* at 0. Moreover, the following result holds.

Theorem 1.1. ([S95, Corollary 1]) *If ξ does not lie in the image $\tau(M)$, then the irreducible representation of highest weight ξ does not occur in $H^0(M, \mathcal{L})$.*

The space $(M \times \overline{\mathcal{O}_\xi^K})_0$ also carries the structure of a complex projective variety. In fact, it is isomorphic to $\text{Proj}(\bigoplus_{k=0}^\infty H^0(M \times \overline{\mathcal{O}_\xi^K}, (\mathcal{L} \boxtimes \overline{\mathcal{L}_\xi^*})^k)^K)$. Moreover, for big enough q , the line bundle $\mathcal{L}^q \boxtimes (\overline{\mathcal{L}_\xi^*})^q$ induces a line bundle $(\mathcal{L}^q \boxtimes (\overline{\mathcal{L}_\xi^*})^q)_0$ over $(M \times \overline{\mathcal{O}_\xi^K})_{ss} // K^\mathbb{C}$, the total space of which is $((\mathcal{L} \boxtimes \overline{\mathcal{L}_\xi^*})^q)|_{(M \times \overline{\mathcal{O}_\xi^K})_{ss}} // K^\mathbb{C}$. For such q , there is an isomorphism

$$H^0(M \times \overline{\mathcal{O}_\xi^K}, (\mathcal{L} \boxtimes \overline{\mathcal{L}_\xi^*})^q)^K \cong H^0((M \times \overline{\mathcal{O}_\xi^K})_0, (\mathcal{L}^q \boxtimes (\overline{\mathcal{L}_\xi^*})^q)_0).$$

Under favourable conditions, *e.g.* the vanishing of all cohomology groups $H^i((M \times \overline{\mathcal{O}_\xi^K})_0, (\mathcal{L}^q \boxtimes (\overline{\mathcal{L}_\xi^*})^q)_0)$, for $i > 0$, the asymptotics of $\dim H^0((M \times$

$\overline{\mathcal{O}_\xi^K})_0, (\mathcal{L}^{kq} \boxtimes (\overline{\mathcal{L}_\xi^*})^{kq})_0$ as $k \rightarrow \infty$ is given by the Riemann-Roch theorem for (singular) complete schemes (cf. [Fu98, Example 18.3.6]).

Although this machinery works well in principle, a major obstruction for applying it in particular cases in order to obtain explicit asymptotic expressions for multiplicities is that the moment maps τ^ξ , and hence their fibres, are in general notoriously hard to compute.

In this paper we are able to compute the moment map $\mu_\mathfrak{k} : X \rightarrow \mathfrak{k}^*$ for the K -action by using an explicit Jordan-theoretic description of X . We also prove that for any $\nu \in \mu_\mathfrak{k}(X)$, the stabilizer, K_ν , of ν acts transitively on the fibre $\mu_\mathfrak{k}^{-1}(\nu)$. As a consequence, the decomposition of $H^0(X, \mathcal{L}^k)$ under K is multiplicity free for every $k \in \mathbb{N}$. We also describe explicitly the moment polytope for $k\mu_\mathfrak{k}$ for any $k \in \mathbb{N}$, i.e., the intersection of $k\mu_\mathfrak{k}(X)$ with a closed Weyl chamber, as well as the integral points in the moment polytope. By Theorem 1.1, these are the only weights that can occur as highest weights of irreducible K -representations in $H^0(X, \mathcal{L}^k)$. We prove that all these integral points in fact do occur. In fact, from the particular form of the integral weights in the moment polytopes for the $k\mu_\mathfrak{k}$ it turns out that it suffices to prove this for $k = 1$, i.e., that all the integral points in the moment polytope for $\mu_\mathfrak{k}$ parameterize irreducible K -representations in $H^0(X, \mathcal{L})$.

In the special case $k = 1$ we prove that the integral points stand in a one-to-one correspondence with the closed $K^\mathbb{C}$ -orbits, X_0, \dots, X_r , in X . The number r is the *rank* of X as a symmetric space. We also give a Jordan-theoretic characterization of these orbits. Using this characterization, we give a geometric decomposition of $H^0(X, \mathcal{L})$ under K . The K -equivariant embedding $W_i \rightarrow H^0(X, \mathcal{L})$ of the irreducible representation W_i corresponding to the orbit X_i is a section for the restriction map $H^0(X, \mathcal{L}) \rightarrow H^0(X_i, \mathcal{L}|_{X_i})$.

We also define an Okounkov body for the line bundle \mathcal{L} and the K -action (cf. [Ok96]) by using a canonical local trivialization of sections. The semigroup defining the Okounkov body describes the initial monomial terms of the polynomials that are local trivializations of highest weight vectors for irreducible K -subrepresentations. Using the decompositions for the spaces $H^0(X, \mathcal{L}^k)$ under K we are able to prove that this semigroup is finitely generated.

The paper is organized as follows. In Section 2 and Section 3 we recall the preliminaries from Lie theory and Jordan theory, respectively. In Section 4 we study the local trivializations of holomorphic sections and use these to define an Okounkov body for \mathcal{L} and the K -action. Section 5 is devoted to closed $K^\mathbb{C}$ -orbits in X and a geometric decomposition theorem. In Section 6 we describe the moment polytope and the symplectic reductions. In Section 7 we prove the decomposition theorem for $H^0(X, \mathcal{L}^k)$ using the results from previous sections. In Section 8 we use the results from Section 7 to identify the Okounkov body with the moment polytope for $\mu_\mathfrak{k}$.

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2. PRELIMINARIES FROM LIE THEORY

Let \mathfrak{g} be a complex simple Lie algebra, and let \mathfrak{h} be a fixed Cartan subalgebra of \mathfrak{g} . Let $\Phi = \Phi(\mathfrak{g}, \mathfrak{h})$ be the set of roots with respect to \mathfrak{h} . Let $\Phi^+ \subseteq \Phi$ be a positive system, and $\{\beta_1, \dots, \beta_s\}$ the corresponding simple roots. For every $\alpha \in \Phi^+$, fix an \mathfrak{sl}_2 -triple $\{E_\alpha, H_\alpha, E_{-\alpha}\}$ with $H_\alpha \in \mathfrak{h}$, $E_\alpha \in \mathfrak{g}_\alpha$, and $E_{-\alpha} \in \mathfrak{g}_{-\alpha}$, normalized so that the identity

$$[E_\alpha, E_\beta] = N_{\alpha, \beta} E_{\alpha+\beta}$$

holds for all $\alpha, \beta \in \Phi$, with constants $N_{\alpha, \beta} \in \mathbb{R}$ satisfying $N_{-\alpha, -\beta} = -N_{\alpha, \beta}$. Also, set $F_\alpha := E_{-\alpha}$ for $\alpha \in \Phi^+$. Let

$$\mathfrak{u} := \bigoplus_{j=1}^s iH_{\beta_j} \oplus \bigoplus_{\alpha \in \Phi^+} \mathbb{R}(E_\alpha - F_\alpha) \oplus \bigoplus_{\alpha \in \Phi^+} \mathbb{R}i(E_\alpha + F_\alpha)$$

be the canonical compact real form of \mathfrak{g} , and $\theta : \mathfrak{g} \rightarrow \mathfrak{g}$ be the associated Cartan involution of \mathfrak{g} . Let

$$\mathfrak{t} := \bigoplus_{j=1}^s iH_{\beta_j} \subseteq \mathfrak{u}$$

be the maximal abelian subalgebra of \mathfrak{u} with $\mathfrak{h}^\theta = \mathfrak{t}$. Define

$$\mathfrak{n}^+ := \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha, \quad \mathfrak{n}^- := \bigoplus_{\alpha \in -\Phi^+} \mathfrak{g}_\alpha,$$

and let

$$\mathfrak{b} := \mathfrak{h} \oplus \mathfrak{n}^+$$

be the Borel subalgebra defined by the positive system Φ^+ . Then \mathfrak{g} can be decomposed as

$$\mathfrak{g} = \mathfrak{b} \oplus \mathfrak{n}^-.$$

For a root, α , let $m_{\beta_i}(\alpha)$ be the multiplicity of β_i in α , i.e., $\alpha = \sum_{i=1}^s m_{\beta_i}(\alpha) \beta_i$.

In this paper we shall be concerned with the special case when \mathfrak{g} admits a simple root that has multiplicity at most one in every positive root. Assume therefore that β_1 is such a simple root. We define subsets of Φ by

$$\Phi_Q := \{\alpha \in \Phi \mid m_{\beta_1}(\alpha) \geq 0\}, \quad \Phi_L := \{\alpha \in \Phi \mid m_{\beta_1}(\alpha) = 0\},$$

and Lie subalgebras

$$\mathfrak{l} := \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi_L} \mathfrak{g}_\alpha, \quad \mathfrak{q} := \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi_Q} \mathfrak{g}_\alpha.$$

We also define

$$\mathfrak{p}^+ := \bigoplus_{\alpha \in \Phi_Q \setminus \Phi_L} \mathfrak{g}_\alpha, \quad \mathfrak{p}^- := \bigoplus_{\alpha \in \Phi_Q \setminus \Phi_L} \mathfrak{g}_{-\alpha}.$$

Then we have

$$\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{p}^+.$$

The Lie algebra \mathfrak{q} is a maximal parabolic subalgebra of \mathfrak{g} containing \mathfrak{b} , and \mathfrak{p}^+ and \mathfrak{l} are the nilpotent radical of \mathfrak{q} and the Levi subalgebra of \mathfrak{q} , respectively. From the assumption that $m_{\beta_1}(\alpha) \in \{-1, 0, 1\}$ for any root α it readily follows that \mathfrak{p}^+ is an abelian subalgebra.

The Lie algebra \mathfrak{l} is reductive with semisimple part

$$\mathfrak{l}' := [\mathfrak{l}, \mathfrak{l}] = \mathfrak{h}_L \oplus \bigoplus_{\alpha \in \Phi_L} \mathfrak{g}_\alpha,$$

where

$$\mathfrak{h}_L := H_{\beta_2} \oplus \cdots \oplus H_{\beta_s}$$

is a Cartan subalgebra of \mathfrak{l}' . Then Φ_L is the set of roots of \mathfrak{l} with respect to \mathfrak{h}_L . The set $\Phi_L^+ := \Phi_L \cap \Phi^+$ is a positive system in Φ_L , and we accordingly define the subalgebras

$$\mathfrak{n}_L^+ := \bigoplus_{\alpha \in \Phi_L^+} \mathfrak{l}_\alpha, \quad \mathfrak{n}_L^- := \bigoplus_{\alpha \in -\Phi_L^+} \mathfrak{l}_\alpha$$

of \mathfrak{l} . The subalgebra \mathfrak{l} is obviously θ -invariant, and we set

$$\mathfrak{k} := \mathfrak{l}^\theta = \mathfrak{l} \cap \mathfrak{u}.$$

Let $\zeta_0 \in \mathfrak{k}$ be a basis vector for $\mathfrak{z}(\mathfrak{l})$, the centre of \mathfrak{l} . Then

$$\mathfrak{l} = \mathfrak{l}' \oplus \mathbb{C}\zeta_0.$$

We now return to the roots in Φ^+ and construct a particular numbering of them. Equip the root lattice with the lexicographic order coming from the identification with the lattice $\mathbb{Z}\beta_1 \oplus \cdots \oplus \mathbb{Z}\beta_s$. Then $\beta_1 > \cdots > \beta_s$. Moreover, $\alpha > \gamma$, for any $\alpha \in \Phi_Q \setminus \Phi_L$ and $\gamma \in \Phi_L$.

We construct a maximal set $\{\gamma_1, \dots, \gamma_r\} \subseteq \Phi_Q \setminus \Phi_L$ of *strongly orthogonal roots*, i.e., the γ_i satisfy the property that for all pairs $\{\gamma_i, \gamma_j\}$ is neither $\gamma_i + \gamma_j$, nor $\gamma_i - \gamma_j$ a root.

First, put $\gamma_1 := \beta_1$. Assuming that $\gamma_1, \dots, \gamma_i$ are defined, let γ_{i+1} be the smallest root in $\Phi_Q \setminus \Phi_L$ such that $\gamma_j \pm \gamma_{i+1} \notin \Phi$ for $j = 1, \dots, i$.

Given the roots $\gamma_1, \dots, \gamma_r$, we now consider a particular decomposition of the the maximal abelian subalgebra $\mathfrak{t} \subseteq \mathfrak{k}$. Let

$$\kappa : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{C}$$

be the Killing form of \mathfrak{g} . Put $\mathfrak{s}' := \mathbb{R}iH_{\gamma_1} \oplus \cdots \oplus \mathbb{R}iH_{\gamma_r} \subseteq \mathfrak{t}$, and consider the decomposition $\mathfrak{t} = \mathfrak{s}' \oplus (\mathfrak{s}')^\perp$, where $(\mathfrak{s}')^\perp \subseteq \mathfrak{t}$ is the orthogonal complement to \mathfrak{s}' with respect to the restriction of κ to \mathfrak{t} . Notice that

$$(\mathfrak{s}')^\perp = \{H \in \mathfrak{t} \mid \gamma_i(H) = 0, \quad i = 1, \dots, r\}.$$

With respect to this decomposition of \mathfrak{t} , let $\zeta \in \mathfrak{t}$ denote the $(\mathfrak{s}')^\perp$ -component of ζ_0 . Now put $\mathfrak{s} := \mathfrak{s}' \oplus \mathbb{R}\zeta$, and consider the decomposition

$$(1) \quad \mathfrak{t} = \mathfrak{s} \oplus (\mathfrak{s})^\perp,$$

where $(\mathfrak{s})^\perp \subseteq \mathfrak{t}$ is the orthogonal complement to \mathfrak{s} with respect to the restriction of κ to \mathfrak{t} .

We now enumerate the roots in $\Phi_Q \setminus \Phi_L$ as follows. First, let $\alpha_1 := \gamma_1, \dots, \alpha_r := \gamma_r$. Then, let $\alpha_{r+1}, \dots, \alpha_n$ be the remaining roots in $\Phi_Q \setminus \Phi_L$, numbered in such a way that

$$\alpha_i < \alpha_j \quad \text{for all } r+1 \leq i < j \leq n$$

(with respect to the lexicographic order above). The vectors $F_1 := F_{\alpha_1}, \dots, F_n := F_{\alpha_n}$ form a basis for \mathfrak{p}^- .

We now shift focus to the level of Lie groups. Let G be a 1-connected complex Lie group with Lie algebra \mathfrak{g} . Define the subgroups $B := N_G(\mathfrak{b})$, $Q := N_G(\mathfrak{q})$, $L := N_G(\mathfrak{p}^-) \cap N_G(\mathfrak{l}) \cap N_G(\mathfrak{p}^+)$ with Lie algebras \mathfrak{b} , \mathfrak{q} , and \mathfrak{l} , respectively. Let $P^- := \langle \exp \mathfrak{p}^- \rangle$ be the integral subgroup of G generated by \mathfrak{p}^- . Moreover, we put $B_L := N_L(\mathfrak{h} \oplus \mathfrak{n}_L^+) \subseteq L$. Then B_L is a Borel subgroup of L . Also, let N_L^+ and N_L^- be the integral subgroups of L with Lie algebras \mathfrak{n}_L^+ and \mathfrak{n}_L^- , respectively.

We will be concerned with the homogeneous space $X := G/Q$. From the point of view of symplectic geometry it is convenient to describe X as a homogeneous space under a compact Lie group. For this purpose, let $U := \exp \langle \mathfrak{u} \rangle \subseteq G$ be the integral subgroup of G generated by \mathfrak{u} . Then, since G/Q is connected, we have

$$X = G/Q = U/K,$$

where

$$K := Q \cap U.$$

Notice that K has Lie algebra \mathfrak{k} .

We will now take a closer look at a particular choice of local coordinates for X . The multiplication map

$$P^- \times Q \rightarrow G, \quad (p, q) \mapsto pq$$

is holomorphic and injective with open image. Moreover the exponential map

$$\exp : \mathfrak{p}^- \rightarrow P^-$$

is a biholomorphic isomorphism. It follows that the map

$$\mathfrak{p}^- \rightarrow P^-Q/Q \subseteq X, \quad z_1 F_1 + \dots + z_n F_n \mapsto \exp(z_1 F_1 + \dots + z_n F_n)Q$$

is an injective holomorphic map with open image.

3. PRELIMINARIES FROM JORDAN THEORY

Recall that there is a one-to-one correspondence between Hermitian symmetric spaces of compact type and semisimple complex Jordan pairs with positive Hermitian involution. In the following we indicate how to obtain a Jordan pair from the Lie algebra \mathfrak{g} . The converse direction is given by the so called Kantor-Koecher-Tits construction, for which we refer to [Be00]. Our main reference is [Lo75, Lo77], and in particular we use the list of Jordan identities in [Lo77] and refer to single identities by JPxy.

Consider the decomposition $\mathfrak{g} = \mathfrak{p}^+ \oplus \mathfrak{l} \oplus \mathfrak{p}^-$. Then, the Lie bracket defines quadratic operators

$$Q^\mp : \mathfrak{p}^\pm \rightarrow \text{Hom}(\mathfrak{p}^\mp, \mathfrak{p}^\pm), \quad x \mapsto Q_x^\pm \quad \text{with} \quad Q_x^\mp(y) := -\frac{1}{2}[[x, y], x].$$

This defines a Jordan pair structure on $(\mathfrak{p}^-, \mathfrak{p}^+)$. For convenience, we set $(V, V') := (\mathfrak{p}^-, \mathfrak{p}^+)$, omit the indices \pm on the quadratic operators and define operators via the relations

$$\{x, y, z\} := D_{x,y}z := Q_{x,z}y := Q_{x+z}y - Q_xy - Q_zy = -[[x, y], z].$$

The context determines the domains of these operators, *e.g.* $\{, , \}$ is a trilinear map from $V \times V' \times V$ to V (resp. from $V' \times V \times V'$ to V'). We also need the *Bergman operator* $B_{x,y}$ which is defined for all pairs $(x, y) \in V \times V'$ by

$$B_{x,y} := \text{Id} - D_{x,y} + Q_xQ_y \in \text{End}(V).$$

The pair (x, y) is called *quasi-invertible* if $B_{x,y}$ is invertible, and then

$$(2) \quad x^y := B_{x,y}^{-1}(x - Q_xy) \in V$$

is called the *quasi-inverse* of (x, y) . In the same way one defines the Bergman operator $B_{y,x} \in \text{End}(V')$ and quasi-inverses $y^x \in V'$ for pairs $(y, x) \in V' \times V$. We note that $B_{x,y}$ is invertible if and only if $B_{y,x}$ is invertible.

The restriction of the Killing form $\kappa : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{C}$ to the product $\mathfrak{p}^- \times \mathfrak{p}^+$ yields a non-degenerate pairing of V and V' , which is given (up to a constant factor) in Jordan theoretic terms by the *trace form*,

$$\tau : V \times V' \rightarrow \mathbb{C}, \quad (x, y) \mapsto \text{Tr } D_{x,y},$$

where Tr denotes the usual trace of linear operators on V . This turns (V, V') into a semisimple complex Jordan pair.

The Cartan involution θ of \mathfrak{g} restricted to \mathfrak{p}^\mp yields antilinear isomorphisms $V \rightleftarrows V'$, which are both denoted by $\bar{x} := \theta(x)$ and that satisfy $Q_{\bar{x}}\bar{y} = \overline{Q_xy}$. Moreover, the map

$$(3) \quad (|) : V \times V \rightarrow \mathbb{C}, \quad (x, z) \mapsto (x|z) := \tau(x, \bar{z})$$

is a positive definite inner product on V , and hence $x \mapsto \bar{x}$ is a *positive Hermitian involution* on (V, V') . By means of this involution, we may identify V with V' .

3.1. Vector fields and group actions. As in (2), we may identify $V = \mathfrak{p}^-$ via the exponential map with an open and dense subset of the compact Hermitian symmetric space $X = G/Q$, i.e., $V \hookrightarrow X$ by $x \mapsto \exp(x)Q$. In this way, the automorphism group G acts on V by birational maps, and elements of its Lie algebra \mathfrak{g} can be identified with vector fields on V , which turn out to be at most quadratic polynomials. Indeed, according to the decomposition $\mathfrak{g} = \mathfrak{p}^+ \oplus \mathfrak{l} \oplus \mathfrak{p}^-$ we have

isomorphisms

$$(4) \quad \begin{aligned} \mathfrak{p}^+ &\cong \{q_v(x) := Q_x v \mid v \in V'\} , \\ \mathfrak{p}^- &\cong \{u(x) := u \mid u \in V\} , \end{aligned}$$

and the vector fields corresponding to \mathfrak{l} are the *derivations* on V , i.e., linear maps $T \in \text{End}(V)$, satisfying

$$T\{x, y, z\} = \{Tx, y, z\} - \{x, T^\#y, z\} + \{x, y, Tz\} ,$$

for all $x, z \in V$, $y \in V'$, where $T^\# \in \text{End}(V')$ is the adjoint map of T with respect to the trace from τ . As an example, for any pair $(x, y) \in V \times V'$, the operator $D_{x,y}$ is a derivation with $(D_{x,y})^\# = D_{y,x}$. Within the context of Jordan theoretic arguments, we identify the Lie algebra \mathfrak{g} with its realization as vector fields on V , so an element $X \in \mathfrak{g}$ is a vector field $\zeta_X : V \rightarrow V$ of the form $\zeta_X(x) = u + Tx + q_v(x)$ with $u \in V$, $T \in \mathfrak{l}$ and $v \in V'$. In order to obtain Lie algebra isomorphisms, we note that the commutator of vector fields $\zeta, \eta \in \mathfrak{g}$ is given by

$$(5) \quad [\zeta, \eta](x) = d\zeta(z) \cdot \eta(z) - d\eta(z) \cdot \zeta(z) ,$$

which differs by sign from the usual convention for the Lie bracket of vector fields. In detail, the commutator of two elements $X_1 = u_1 + T_1 + q_{v_1}$ and $X_2 = u_2 + T_2 + q_{v_2}$ is given by

$$[X_1, X_2] = (T_1 u_2 - T_2 u_1) + (D_{u_2, v_1} + [T_1, T_2] - D_{u_1, v_2}) + (q_{T_2^\# v_1} - q_{T_1^\# v_2}) ,$$

and the Killing form κ on \mathfrak{g} translates to

$$(6) \quad \kappa(X_1, X_2) = \kappa_{\mathfrak{l}}(T_1, T_2) + 2 \text{Tr}(T_1 T_2) - 2\tau(u_1, v_2) - 2\tau(u_2, v_1) ,$$

where $\kappa_{\mathfrak{l}}$ denotes the Killing form on \mathfrak{l} . For both formulas, see *e.g.* [Sa80, §7].

The birational group action of $\exp(\mathfrak{p}^\pm)$ on $V \subseteq X$ is given and denoted by

$$\begin{aligned} t_u(x) &:= \exp(u)x = x + u && \text{for } u \in V \text{ (translation)}, \\ \tilde{t}_v(x) &:= \exp(q_v)x = x^v && \text{for } v \in V' \text{ (quasi-translation)}. \end{aligned}$$

The subgroup $L \subseteq G$ is identified with the identity component of the automorphism group $\text{Aut}(V, V')$ of the Jordan pair, which consists of linear automorphisms $h \in \text{GL}(V)$ satisfying $h\{x, y, z\} = \{hx, h^{-\#}y, hz\}$ for all $x, z \in V$, $y \in V'$, where $h^{-\#} := (h^\#)^{-1}$ and $h^\#$ is the adjoint map of h with respect to the trace form τ . Therefore, $h \in L$ acts on $V \subseteq X$ by linear transformations $x \mapsto hx$. As an example, for quasi-invertible pairs $(x, y) \in V \times V'$, the Bergman operator $B_{x,y}$ is a Jordan pair automorphism with $(B_{x,y})^\# = B_{y,x}$.

The Cartan involution θ corresponding to the compact real forms $\mathfrak{k} \subseteq \mathfrak{l}$ and $\mathfrak{u} \subseteq \mathfrak{g}$ translates to $\theta(u + T + q_v) = \bar{v} - T^* + q_{\bar{u}}$, where T^* denotes the adjoint of T with respect to the inner product (3) on V . Therefore,

$$\mathfrak{k} = \{T \in \mathfrak{l} \mid T = -T^*\} , \quad \mathfrak{u} = \{u + T + q_{\bar{u}} \mid u \in V, T \in \mathfrak{k}\} .$$

In addition, we note that the centre of \mathfrak{k} is given by $\mathfrak{z}(\mathfrak{k}) = \mathbb{R}(i \text{Id})$, and the Lie group K is the connected component of the group of automorphisms $h \in \text{Aut}(V, V')$ satisfying $h = h^{-*}$, i.e., unitary automorphisms.

3.2. Idempotents, Peirce decomposition, and rank. An *idempotent* is a pair $\mathbf{e} = (e, e') \in V \times V'$ satisfying the relations $Q_e e' = e$ and $Q_{e'} e = e'$. Let $\mathcal{I} \subset V \times V'$ denote the set of idempotents. For $\mathbf{e} \in \mathcal{I}$, the operators $D_{e, e'} \in \text{End}(V)$ and $D_{e', e} \in \text{End}(V')$ are diagonalizable with spectra in $\{0, 1, 2\}$, and the decomposition into eigenspaces,

$$V = V_2(\mathbf{e}) \oplus V_1(\mathbf{e}) \oplus V_0(\mathbf{e}), \quad V' = V'_2(\mathbf{e}) \oplus V'_1(\mathbf{e}) \oplus V'_0(\mathbf{e}),$$

is called the *Peirce decomposition* with respect to \mathbf{e} . We note that in general, $V'_k(\mathbf{e})$ differs from the image of $V_k(\mathbf{e})$ under the involution of (V, V') , i.e., $V'_k(\mathbf{e}) \neq \overline{V_k(\mathbf{e})}$. The *Peirce spaces* $V_k := V_k(\mathbf{e})$, $V'_k := V'_k(\mathbf{e})$ are subject to the following multiplication rules (the *Peirce rules*)

$$\{V_i, V'_j, V_k\} \subseteq V_{i-j+k}, \quad \{V_2, V'_0, V\} = \{V_0, V'_2, V\} = \{0\},$$

where $V_\ell = \{0\}$ and $V'_\ell = \{0\}$ if $\ell \notin \{0, 1, 2\}$. In particular, (V_k, V'_k) is a subpair of (V, V') . Two idempotents $\mathbf{e} = (e, e')$, $\mathbf{c} = (c, c')$ are (*strongly*) *orthogonal* if $c \in V_0(\mathbf{e})$ or equivalently $e \in V_0(\mathbf{c})$. In this case, the sum $\mathbf{e} + \mathbf{c} = (e + c, e' + c')$ is also an idempotent. A non-zero idempotent is called *primitive*, if it is not the sum of two orthogonal non-zero idempotents. A *frame of idempotents* $(\mathbf{e}_1, \dots, \mathbf{e}_r)$ is a maximal system of primitive orthogonal idempotents. The length r of a frame of idempotents is an invariant of the Jordan pair (V, V') , called the *rank* and denoted by $\text{rk } V := r$.

For a system of (pairwise) orthogonal idempotents $(\mathbf{e}_1, \dots, \mathbf{e}_k)$, so in particular for a frame, the operators $(D_{e_\ell, e'_\ell})_{\ell=1, \dots, k}$ form a commuting set of diagonalizable operators, and hence induce the *joint Peirce decomposition*

$$V = \bigoplus_{0 \leq i \leq j \leq k} V_{ij} \quad \text{with} \quad V_{ij} = \{x \in V \mid \{e_\ell, e'_\ell, x\} = (\delta_{i\ell} + \delta_{j\ell})x \text{ for all } \ell\},$$

and likewise for $(D_{e'_\ell, e_\ell})_{\ell=1, \dots, k}$ and V' . Setting $V_{ji} := V_{ij}$ and $V'_{ji} := V'_{ij}$ for $i \neq j$, the Peirce rules refine to the *joint Peirce rules*

$$\{V_{ij}, V'_{jk}, V_{kl}\} \subseteq V_{i\ell},$$

and all other types of products vanish. Again, we point out that the image of V_{ij} under the involution of (V, V') in general differs from V'_{ij} , unless we consider a special class of idempotents, namely those defined by tripotents, which we discuss in the next section.

If (V, V') is simple of rank r , then, for any primitive idempotent $\mathbf{e} \in \mathcal{I}$, we set

$$(7) \quad p := \tau(e, e') = 2 + \dim V_1(\mathbf{e})$$

which is independent of the choice of \mathbf{e} . This structure constant of (V, V') appears in subsequent formulas which involve the Killing form of \mathfrak{g} .

We also need the notion of *rank* for arbitrary elements $x \in V$ or $y \in V'$. For $x \in V$ the subspace $[x] := Q_x V' \subseteq V$ is called the *principal inner ideal* generated by x . The *rank* of x , $\text{rk } x$, is defined as the maximum length of all chains $[x_0] \subsetneq [x_1] \subsetneq \cdots \subsetneq [x_k]$ with $x_i \in [x]$. Similarly, one defines $[y] := Q_y V \subseteq V'$ and $\text{rk } y$ for $y \in V'$. If $\mathbf{e} = (e, e')$ is an idempotent, then the Peirce rules imply $[e] = V_2(\mathbf{e})$ and $[e'] = V'_2(\mathbf{e})$, and it turns out [Lo91, §3], that $\text{rk } e = \text{rk } e'$. We therefore define $\text{rk } \mathbf{e} := \text{rk } e$ and call this the *rank of the idempotent* \mathbf{e} . The set \mathcal{I} of idempotents therefore decomposes into subset of constant rank idempotents, denoted by $\mathcal{I}_k := \{\mathbf{e} \in \mathcal{I} \mid \text{rk } \mathbf{e} = k\}$. Since we assume (V, V') to be finite dimensional and simple, each element $e \in V$ admits a *completion to an idempotent*, i.e., an element $e' \in V'$ such that $(e, e') \in \mathcal{I}$. From this, it follows that the decomposition of an idempotent \mathbf{e} of rank k into primitive orthogonal idempotents has exactly k summands. Therefore, the maximum of all ranks of elements in V coincides with the rank of V as it is defined above.

3.3. Tripotents and spectral decomposition. The involution on (V, V') admits the definition of (odd) powers of elements, namely for $x \in V$ define $x^{(1)} := x$ and inductively $x^{(2k+1)} := Q_x \overline{x^{(2k-1)}}$ for $k \geq 1$. An element $e \in V$ is called a *tripotent*, if $e^{(3)} = e$, i.e., $e = Q_e \bar{e}$. Equivalently, e is a tripotent if and only if (e, \bar{e}) is an idempotent. In particular, all notions defined in the last section apply to the idempotent (e, \bar{e}) . Without causing ambiguities in notation, we may identify e with (e, \bar{e}) if necessary. Concerning the Peirce decomposition, we note that $V'_k(e) = \overline{V_k(e)}$ for a tripotent e , and $V'_{ij} = \overline{V_{ij}}$ for a system of orthogonal tripotents (e_1, \dots, e_k) . The set of tripotents is denoted by $S \subseteq V$.

For the explicit description of the moment map on X , we will make use of the following spectral theorem [Lo77, §3.12].

Theorem 3.1 (Spectral decomposition). *Let (V, V') be a finite dimensional semisimple Jordan pair with positive Hermitian involution. Then every element $x \in V$ admits a unique decomposition*

$$x = \sigma_1 e_1 + \cdots + \sigma_k e_k, \quad \sigma_1 > \cdots > \sigma_k > 0,$$

where the e_i are pairwise orthogonal non-zero tripotents which are real linear combinations of powers of x , and $\sigma_i \in \mathbb{R}$.

3.4. Idempotents and roots. Idempotents are related to \mathfrak{sl}_2 -triples in \mathfrak{g} in the following way: If $(e, e') \in \mathcal{I}$ is an idempotent, then $(E, H, F) := (q_{e'}, D_{e, e'}, -e)$ is an \mathfrak{sl}_2 -triple in \mathfrak{g} with $q_{e'} \in \mathfrak{p}^+$, $D_{e, e'} \in \mathfrak{l}$ and $-e \in \mathfrak{p}^-$ (constant vector field). Indeed, according to (5) and using JP12, it follows

that

$$\begin{aligned} [D_{e,e'}, q_{e'}] &= q_{D_{e'}, e'} = 2q_{e'}, \\ [D_{e,e'}, -e] &= -D_{e,e'}e = -2e, \\ [-e, q_{e'}] &= D_{e,e'}. \end{aligned}$$

Conversely, if (E, H, F) is an \mathfrak{sl}_2 -triple in \mathfrak{g} with $E \in \mathfrak{p}^+$, $H \in \mathfrak{l}$ and $F \in \mathfrak{p}^-$, then the identifications $V' \cong \mathfrak{p}^+$ and $V \cong \mathfrak{p}^-$ in (4) yield an idempotent $(e, e') \in \mathcal{I}$ with $E = q_{e'}$, $F = -e$ and $H = D_{e,e'}$.

This correspondence between idempotents and certain \mathfrak{sl}_2 -triples in \mathfrak{g} also applies to tripotents: Starting with a tripotent $e \in S$, we obtain the \mathfrak{sl}_2 -triple $(E, H, F) := (q_{\bar{e}}, D_{e, \bar{e}}, -e)$ with the additional property $\theta(E) = -F$, where θ denotes the Cartan involution on \mathfrak{g} , and hence $iH \in \mathfrak{k}$. Conversely, any \mathfrak{sl}_2 -triple (E, H, F) with $E \in \mathfrak{p}^+$ and $\theta(E) = -F$ yields a tripotent $e \in S$ corresponding to $E = q_{\bar{e}}$.

In particular, the \mathfrak{sl}_2 -triples associated to the system of strongly orthogonal roots $\gamma_1, \dots, \gamma_r$ yield a system (e_1, \dots, e_r) of tripotents, and it is straightforward to see that strong orthogonality of the roots is equivalent to strong orthogonality of the tripotents. Therefore, (e_1, \dots, e_r) is a frame associated to the system of strongly orthogonal roots. We summarize the situation by

$$(E_{\gamma_j}, H_{\gamma_j}, F_{\gamma_j}) = (q_{\bar{e}_j}, D_{e_j, \bar{e}_j}, -e_j).$$

Moreover, using the Killing form (6), the relation $\gamma_j = c \cdot \kappa(H_{\gamma_j}, -)$ with $c = 2/\kappa(H_{\gamma_j}, H_{\gamma_j})$ yields

$$(8) \quad \gamma_j(T) = \frac{1}{p} \tau(Te_j, \bar{e}_j) \quad \text{for all } T \in \mathfrak{l},$$

where p is the structure constant defined by (7). This is the Jordan theoretic description of the strongly orthogonal roots.

Recall the decomposition (1) of the maximal abelian subalgebra \mathfrak{t} of \mathfrak{k} into $\mathfrak{t} = \mathfrak{s} \oplus \mathfrak{s}^\perp$. In Jordan theoretic terms, we obtain

$$\mathfrak{s} = \langle i D_{e_j, \bar{e}_j}, i \text{Id}_V \mid j = 1, \dots, r \rangle_{\mathbb{R}},$$

and the condition $\mathfrak{s}^\perp \subseteq \{T \in \mathfrak{l} \mid \gamma_j(T) = 0\}$ translates to

$$\mathfrak{s}^\perp \subseteq \{T \in \mathfrak{k} \mid Te_j = 0 \text{ for all } j\}.$$

3.5. Determinants. We have to deal with two kinds of determinants. On the one hand, there is the *Jordan pair determinant* $\Delta : V \times V' \rightarrow \mathbb{C}$ associated to a Jordan pair (V, V') , often also called the *generic minimum polynomial*, cf. [Lo75, §16]. On the other hand, let $\mathbf{e} = (e, e')$ be an idempotent in (V, V') , then the Peirce 2-space $V_2(e)$ becomes a unital Jordan algebra with product $x \circ z := \frac{1}{2} \{x, e', z\}$ and unit element e . The Jordan algebra determinant corresponding to $V_2(e)$ is denoted by $\Delta_{\mathbf{e}} : V_2(\mathbf{e}) \rightarrow \mathbb{C}$. Likewise, $V'_2(\mathbf{e})$ is a Jordan algebra with product $y \circ w := \frac{1}{2} \{y, e, w\}$, unit element e' , and Jordan algebra determinant

$\Delta'_e : V'_2(\mathbf{e}) \rightarrow \mathbb{C}$. The connection between Jordan pair determinant and Jordan algebra determinants is given as follows:

Lemma 3.2. *If $\mathbf{e} = (e, e')$ is an idempotent, then*

$$(9) \quad \Delta_{\mathbf{e}}(x) = \Delta(e - x, e'), \quad \Delta'_{\mathbf{e}}(y) = \Delta(e, e' - y)$$

for all $x \in V_2(\mathbf{e})$ and $y \in V'_2(\mathbf{e})$.

Proof. Each identity is a consequence of the relation between Jordan algebra inverses and quasi-inverses, e.g. $x^{-1} = e + (e - x)^{e'}$ for invertible $x \in V_2(\mathbf{e})$. We omit the details. \square

In the following, we use (9) to extend the Jordan algebra determinants $\Delta_{\mathbf{e}}$ and $\Delta'_{\mathbf{e}}$ to polynomial maps on V and V' . By abuse of notation, these extensions are also denoted by $\Delta_{\mathbf{e}}$ and $\Delta'_{\mathbf{e}}$. We note that if $V = V_2(\mathbf{e}) \oplus V_1(\mathbf{e}) \oplus V_0(\mathbf{e})$ is the Peirce decomposition with respect to \mathbf{e} , then $\Delta_{\mathbf{e}}$ vanishes on $V_1(\mathbf{e}) \oplus V_0(\mathbf{e})$. Analog results hold for $\Delta'_{\mathbf{e}}$. For later use, we note the following relation between the rank of idempotents and zeros of Jordan algebra determinants:

Lemma 3.3. *Let $x \in V$ be a fixed element, and $k \in \mathbb{N}$. Then, $\Delta_{\mathbf{c}}(x) = 0$ for all idempotents \mathbf{c} of rank k if and only if $k > \text{rk } x$. The same holds for $y \in V'$ and $\Delta'_{\mathbf{c}}(y) = 0$ in place of $\Delta_{\mathbf{c}}(x) = 0$.*

Proof. First assume that $k \leq \text{rk } x$. Let $\mathbf{e} = (e, e')$ be a completion of $e = x$ to an idempotent, and let $\mathbf{e} = \mathbf{e}_1 + \dots + \mathbf{e}_{\ell}$ be a decomposition into primitive orthogonal idempotents. Then, $\ell = \text{rk } \mathbf{e} = \text{rk } x \geq k$, and hence $\mathbf{c} := \mathbf{e}_1 + \dots + \mathbf{e}_k$ is well-defined and satisfies $\Delta_{\mathbf{c}}(x) = \Delta_{\mathbf{c}}(e) = 1$. This proves the 'only if' part. For the converse direction assume $\Delta_{\mathbf{c}}(x) \neq 0$ for some $\mathbf{c} \in \mathcal{I}_k$. Let $x = x_2 + x_1 + x_0$ be the components of x in the Peirce decomposition of V with respect to \mathbf{c} . Then, $\Delta_{\mathbf{c}}(x) = \Delta_{\mathbf{c}}(x_2)$, and this is non-vanishing if and only if x_2 is invertible in the unital Jordan algebra $V_2(\mathbf{c})$. Furthermore, this is equivalent to the identity $[x_2] = V_2(\mathbf{c})$, where $[x_2] = Q_{x_2}V'$ is the principal inner ideal corresponding to x_2 . Since $V_2(\mathbf{c}) = [c]$, this implies that $\text{rk } x_2 = \text{rk } c = \text{rk } \mathbf{c}$. Now the statement follows from the inequality $\text{rk } x_2 \leq \text{rk } x$, cf. [Lo91, §3]. \square

3.6. Jordan theoretic model of X . The concept of quasi-inverses (2) provides a Jordan theoretic model for the compact Hermitian symmetric space $X = G/Q$ (due to O. Loos [Lo77]): For each $a \in V'$ the map $\iota_a : V \rightarrow X$ given by $\iota_a(x) = \exp(q_a) \exp(x)Q = \tilde{t}_a t_x Q$ is an open and dense imbedding of V in X . This yields an open covering of X by the subsets $X_a := \iota_a(V)$, $a \in V'$. It turns out that $X_a \cap X_b$ is the image of $\{x \in V \mid (x, a - b) \text{ quasi-invertible}\}$ under ι_a , and the transition map $\varphi_b^a = \iota_b^{-1} \circ \iota_a$ is given by

$$\varphi_b^a(x) = x^{a-b}.$$

This description of X may be summarized in the equivalence relation $X \cong (V \times V')/R$ with

$$(10) \quad (x, a) R(\tilde{x}, b) \iff \begin{cases} (x, a - b) \text{ is quasi-invertible} \\ \text{and } \tilde{x} = x^{a-b}. \end{cases}$$

The equivalence class of an element (x, a) is denoted by $[x : a]$. For $a = 0$, the imbedding ι_0 is just the standard imbedding of $V \cong \mathfrak{p}^-$ into X , which we have already used in Section 3.1. In the following, we write $V \subseteq X$ for the identification of V with its image under ι_0 in X .

Some questions require yet another description of the elements of X . Consider the description via equivalence classes as it is given in (10). Whereas the chart maps concerns elements with fixed second entry $a \in V'$, i.e., $x \mapsto [x : a]$, the following proposition selects for each element in X a representative which is adapted to arguments concerning the action of the automorphism group of (V, V') on X .

Proposition 3.4. *Let $[x : a]$ be an element of $X = (V \times V')/R$. Then, there exist an idempotent $\mathbf{e} = (e, e') \in \mathcal{I}$ and an element $z \in V_0(\mathbf{e})$ such that $[x : a] = [e + z : e']$. Moreover, the idempotent \mathbf{e} can be chosen to be tripotent, i.e., $\mathbf{e} = (e, \bar{e})$.*

Proof. The existence of a representative for $[x : a]$ of the form $(e + z, e')$ with idempotent \mathbf{e} and $z \in V_0(\mathbf{e})$ is proved in [Lo94], see Theorem 3.8, Proposition 4.6, and Theorem 4.7 therein. The possibility to choose \mathbf{e} to be tripotent, follows from the fact that in our setting $((V, V')$ being finite dimensional and simple), for any idempotent \mathbf{e} there exists a tripotent c such that $V_2(\mathbf{e}) = V_2(c)$. Therefore, we may assume that the idempotent $e = (e_+, e_-)$ used in the last part of Proposition 6.5 in [Lo94] is in fact a tripotent. \square

4. LINE BUNDLES AND REPRESENTATION SPACES FOR G

4.1. Prequantum bundles on G/Q . Let $-\lambda \in \mathfrak{h}^*$ be the fundamental weight associated to the simple root β_1 . Let $\chi_\lambda : Q \rightarrow \mathbb{C}^\times$ be the holomorphic character determined by the condition $d\chi_\lambda(e)|_{\mathfrak{h}} = \lambda$, and let $\mathcal{L} := G \times_Q \mathbb{C} \rightarrow X$ be the line bundle associated to χ_λ . We shall be concerned with the family $H^0(X, \mathcal{L}^k)$, for $k \in \mathbb{N}$, of G -representations, and with the decomposition under the group L . Notice that K is a maximal compact subgroup of L , so that the decompositions under L amount to the decompositions under K .

In order to realize the homogeneous space X as a coadjoint U -orbit in \mathfrak{u}^* we extend the \mathbb{C} -linear functional $\lambda \in \mathfrak{h}^*$ to a functional on \mathfrak{g} by requiring that it annihilate all root spaces \mathfrak{g}_α . This convention will hereafter be used for extending linear functionals on \mathfrak{h} to linear functionals on \mathfrak{g} . Similarly we extend \mathbb{R} -linear functionals on \mathfrak{t} to \mathbb{R} -linear functionals on \mathfrak{u} .

Notice that λ has imaginary values on \mathfrak{t} , so that $i\lambda$ restricts to a real-valued \mathbb{R} -linear functional on \mathfrak{t} . We will write $i\lambda$ instead of $i\lambda|_{\mathfrak{u}}$ for the induced \mathbb{R} -linear functional on \mathfrak{u} . Using the above conventions, X can be realized as the coadjoint orbit, $\mathcal{O}_\lambda \subseteq \mathfrak{u}^*$, of $i\lambda$. When \mathcal{O}_λ is equipped with the Kostant-Kirillov symplectic form, ω_λ , the action of U is Hamiltonian with moment map

$$\mu : \mathcal{O}_\lambda \rightarrow \mathfrak{u}^*$$

being the inclusion $\mathcal{O}_\lambda \subseteq \mathfrak{u}^*$. The action of the subgroup K is also Hamiltonian, with moment map

$$\mu_{\mathfrak{k}} : \mathcal{O}_\lambda \rightarrow \mathfrak{k}^*, \quad \mu_{\mathfrak{k}}(f)(x) := f(x), \quad f \in \mathcal{O}_\lambda, \quad x \in \mathfrak{k}.$$

The line bundle \mathcal{L} is a prequantum line bundle for the Kostant-Kirillov form on \mathcal{O}_λ . In order to interpret the tensor powers \mathcal{L}^k as prequantum line bundles on $X \cong \mathcal{O}_\lambda$, we use the natural isomorphism $\mathcal{O}_\lambda \cong \mathcal{O}_{k\lambda}, f \mapsto kf$, to identify the symplectic manifolds $(\mathcal{O}_{k\lambda}, \omega_{k\lambda})$ and $(\mathcal{O}_\lambda, k\omega_\lambda)$. Then \mathcal{L}^k is a prequantum line bundle for $(\mathcal{O}_\lambda, k\omega_\lambda)$. The moment map for the K -action on $(\mathcal{O}_\lambda, k\omega_\lambda)$ is

$$\mu_{\mathfrak{k}}^k := k\mu_{\mathfrak{k}}.$$

4.2. Trivialization of sections. Consider the dual space $H^0(X, \mathcal{L})^*$, which is a highest weight module of highest weight $-\lambda$. We make a specific choice of a highest weight vector. For this purpose we identify the space of holomorphic sections $H^0(X, \mathcal{L})$ with the space of Q -equivariant holomorphic functions $F : G \rightarrow \mathbb{C}$ having the Q -equivariance property

$$F(gq) = \chi_\lambda^{-1}(q)F(g), \quad g \in G, \quad q \in Q.$$

Define $w_1 \in H^0(X, \mathcal{L})^*$ as the linear functional $F \mapsto \text{ev}_e(F) := F(e)$. Then w_1 is a cyclic vector for $H^0(X, \mathcal{L})^*$ as a $U(\mathfrak{g})$ -module, as well as a $U(\mathfrak{p}^-)$ -module. For $k \in \mathbb{N}$, the vector $w_k := w_1^{\otimes k} \in H^0(X, \mathcal{L}^k)^*$ is then a highest weight vector for $H^0(X, \mathcal{L}^k)^*$. Given these normalizations of highest weight vectors of the spaces $H^0(X, \mathcal{L}^k)^*$ we now consider local trivializations of the bundles \mathcal{L}^k .

The principal bundle $q : G \rightarrow G/Q$ is trivial over the open set P^-Q/Q . Hence the bundles \mathcal{L}^k , being associated to this principal bundle, are also trivial over P^-Q/Q . If we identify a section φ of \mathcal{L}^k with a linear functional on the dual space $H^0(X, \mathcal{L}^k)^*$, then the realization of φ as a Q -equivariant holomorphic function, F , on G is given by

$$F(g) = \varphi(g.w_k).$$

The trivialization of a section, viewed as a Q -equivariant holomorphic function $F : G \rightarrow \mathbb{C}$, is given by the restriction of F to P^- . In the local coordinates (2), the restriction of F to P^- is given by the function

$$(11) \quad f(z_1, \dots, z_n) = \varphi(\exp(z_1 F_1 + \dots + z_n F_n).w_k), \quad z \in \mathfrak{p}^-.$$

Proposition 4.1. *Let $s \in H^0(X, \mathcal{L}^k)$ be a weight vector of weight δ . If scalar multiples of the two monomial terms z^a and z^b occur in the polynomial that trivializes s , then the identity*

$$k\lambda + \sum_{j=1}^n a_j \alpha_j = k\lambda + \sum_{j=1}^n b_j \alpha_j.$$

holds in the weight lattice. Moreover, $\delta = k\lambda + \sum_{j=1}^n a_j \alpha_j$.

Proof. The holomorphic section s corresponds to a linear functional φ on the space $H^0(X, \mathcal{L}^k)^*$. The trivialization of s is then given by the polynomial function

$$\begin{aligned} f(z_1, \dots, z_n) &= \varphi(\exp(z_1 F_1 + \dots + z_n F_n).w_k) \\ &= \sum_{c \in \mathbb{N}_0^n} \frac{1}{c_1! \dots c_n!} z^c \varphi(F^c.w_k), \end{aligned}$$

where $F^c := F_1^{c_1} \dots F_n^{c_n}$. Since each vector $F^c.w_k$ is a weight vector (of weight $-k\lambda - \sum_{j=1}^n c_j \alpha_j$), the functional φ can only have nonzero values on vectors $F^c.w_k$ of the fixed weight $-\delta$. \square

We now define a *valuation-like function*

$$v : \bigsqcup_{k \in \mathbb{N}} H^0(X, \mathcal{L}^k) \setminus \{0\} \rightarrow \mathbb{N}_0^n,$$

i.e., a function satisfying the properties

$$\begin{aligned} v(st) &= v(s) + v(t), \quad s \in H^0(X, \mathcal{L}^k) \setminus \{0\}, t \in H^0(X, \mathcal{L}^\ell) \setminus \{0\}, \\ v(s+t) &\geq \min(v(s), v(t)), \quad s, t \in H^0(X, \mathcal{L}^k) \setminus \{0\}, s+t \neq 0, \\ v(\lambda s) &= v(s), \quad s \in H^0(X, \mathcal{L}^k) \setminus \{0\}, \lambda \in \mathbb{C}^\times. \end{aligned}$$

If $s \in H^0(X, \mathcal{L}^k) \setminus \{0\}$ is trivialized as the polynomial $f = \sum_{a \in \mathbb{N}_0^n} c_a z^a$, let

$$(12) \quad v(s) := \min\{a \in \mathbb{N}_0^n \mid c_a \neq 0\},$$

where the minimum refers to the inverse lexicographic order on \mathbb{Z}^n .

Remark 4.2. The function v , although it seems to depend on the particular local coordinates chosen, has a global geometric meaning. In fact, v can be interpreted as defining the “successive orders of vanishing” of s along a flag of irreducible subvarieties $X_0 \subseteq \dots \subseteq X_{n-1} \subseteq X_n = X$, where $\dim X_i = i$, in the setting defined by Okounkov ([Ok96]), and later developed by Lazarsfeld-Mustață ([LM09]). This interpretation will however not play any role in the rest of this paper. Instead, the particular local expression (12) will be useful.

To the function v we attach the semigroup

$$(13) \quad S(\mathcal{L}, N_L^+, v) := \{(k, v(s)) \mid s \in H^0(X, \mathcal{L}^k)^{N_L^+} \setminus \{0\}\} \subseteq \mathbb{N} \times \mathbb{N}_0^n,$$

and the closed convex cone, $C(\mathcal{L}, N_L^+, v)$, in $\mathbb{R} \times \mathbb{R}^n$ which is generated by the semigroup $S(\mathcal{L}, N_L^+, v)$. Finally, we define the *Okounkov body*,

$$(14) \quad \Delta(\mathcal{L}, N_L^+, v) := C(\mathcal{L}, N_L^+, v) \cap (\{1\} \times \mathbb{R}^n).$$

5. L -TYPES AND L -ORBITS IN X

Recall that L is the identity component of the automorphism group of the Jordan pair (V, V') . We give a geometric proof of the decomposition of $H^0(X, \mathcal{L})$ into irreducible L -modules. More precisely, we claim that the decomposition is obtained by restriction to the closed L -orbits in $X = G/Q$.

5.1. Closed L -orbits. Recall that the set $\mathcal{I} \subseteq V \times V'$ of idempotents decomposes into the subsets of constant rank idempotents,

$$\mathcal{I} = \bigcup_{k=0}^r \mathcal{I}_k \quad \text{with} \quad \mathcal{I}_k = \{\mathbf{e} \in \mathcal{I} \mid \text{rk } \mathbf{e} = k\},$$

where r denotes the rank of the Jordan pair (V, V') . Assuming that (V, V') is simple (or equivalently, that the Hermitian symmetric space X is irreducible), it is well-known that the \mathcal{I}_k are the connected components of \mathcal{I} (with respect to the induced topology from $V \times V'$), and that the L -action on $V \times V'$ restricts to a transitive L -action on each component $\mathcal{I}_k \subseteq V \times V'$, cf. [Lo75, §17.1]. Let N_k be the dimension of the Peirce 2-space $\dim V_2(\mathbf{e})$ for some idempotent $\mathbf{e} \in \mathcal{I}_k$, which is independent of the choice of \mathbf{e} , and let $\text{Gr}_{N_k}(V)$ be the (classical) Grassmannian manifold of N_k -dimensional subspaces in V , equipped with the natural L -action induced by the L -action on V . Consider the subset

$$\mathcal{P}_k := \{U \subseteq V \mid U = V_2(\mathbf{e}) \text{ for some } \mathbf{e} \in \mathcal{I}_k\} \subseteq \text{Gr}_{N_k}(V).$$

Since $V_2(h\mathbf{e}) = hV_2(\mathbf{e})$ for any $h \in L$ and $\mathbf{e} \in \mathcal{I}_k$, the map

$$\pi_k : \mathcal{I}_k \rightarrow \mathcal{P}_k, \quad \mathbf{e} \mapsto V_2(\mathbf{e})$$

is L -equivariant, and hence \mathcal{P}_k is an L -orbit in $\text{Gr}_{N_k}(V)$. In particular, it is a smooth projective variety, called the *Peirce variety* of rank k . We call two idempotents $\mathbf{e}, \mathbf{c} \in \mathcal{I}_k$ *Peirce equivalent*, if $\pi_k(\mathbf{e}) = \pi_k(\mathbf{c})$, i.e., if their Peirce 2-spaces in V coincide.

We show that the Peirce varieties \mathcal{P}_k can be imbedded L -equivariantly into the compact Hermitian symmetric space $X = G/Q$. Recall the Jordan-theoretic description of X via *projective equivalence*, $X \cong (V \times V')/R$. The following is a crucial observation due to O. Loos, cf. [Lo94, §2.6].

Lemma 5.1. *Two idempotents $\mathbf{e}, \mathbf{c} \in \mathcal{I}_k \subseteq V \times V'$ are Peirce equivalent if and only if they are projectively equivalent, i.e., $V_2(\mathbf{e}) = V_2(\mathbf{c})$ if and only if $[e : e'] = [c : c']$.*

Applying this lemma, it follows that the map

$$\iota_k : \mathcal{P}_k \rightarrow X, \quad U \mapsto [e : e'] \quad \text{for} \quad U = V_2(\mathbf{e}) \quad \text{with} \quad \mathbf{e} = (e, e')$$

is well-defined and one-to-one. We may illustrate the situation in the following commutative diagram

$$(15) \quad \begin{array}{ccc} \mathcal{I}_k & \xrightarrow{\text{id}} & V \times V' \\ \pi_k \downarrow & & \downarrow \pi \\ \mathcal{P}_k & \xrightarrow{\iota_k} & X \end{array} .$$

Moreover, since $h\mathbf{e} = (he, h^{-\#}e')$ and $h[e : e'] = [he : h^{-\#}e']$, this diagram is also L -equivariant, and in particular ι_k is an L -equivariant isomorphism onto its image

$$X_k := \iota_k(\mathcal{P}_k) \subseteq X ,$$

which is a closed L -orbit in X . In this way, we have identified $r + 1$ different L -orbits in X . Recalling a classical result, that there are precisely $r + 1$ closed L -orbits in X , we summarize the arguments of this section.

Proposition 5.2. *The closed L -orbits in X are precisely the complex analytic submanifolds*

$$X_k = \{[e : e'] \mid (e, e') \in \mathcal{I}_k\} \subseteq X$$

for $k = 0, \dots, r$. The L -orbit X_k is L -equivariantly isomorphic via ι_k to the Peirce variety \mathcal{P}_k .

5.2. Restriction isomorphism. In the following, we use the notation of diagram (15) and consider the restrictions of the fundamental line bundle \mathcal{L} on X to the closed L -orbits $(X_k)_{k=0, \dots, r}$ that are described in Proposition 5.2. Due to [Lo78, Proposition 3.1], the projection π trivializes the line bundle \mathcal{L} , i.e., $\pi^*\mathcal{L}$ is trivial, and sections in \mathcal{L} can be identified with holomorphic maps on $V \times V'$ satisfying a certain cocycle condition, i.e.,

$$H^0(X, \mathcal{L}) \cong \left\{ f : V \times V' \rightarrow \mathbb{C} \text{ hol.} \left| \begin{array}{l} f(x, y) = \Delta(x, y - y') \cdot f(x^{y-y'}, y') \\ \text{for all quasi-invertible } (x, y - y') \end{array} \right. \right\} .$$

Let $\mathcal{L}|_{X_k}$ denote the restriction of the line bundle to the closed L -orbit $X_k \subseteq X$. By compactness of X_k , the space $H^0(X_k, \mathcal{L}|_{X_k})$ is finite dimensional. Moreover, L acts irreducibly. Due to Proposition 5.2 and the commutativity of (15), we obtain the identifications

$$\begin{aligned} H^0(X_k, \mathcal{L}|_{X_k}) &\cong H^0(\mathcal{P}_k, \iota_k^*\mathcal{L}) \\ &\cong \left\{ f : \mathcal{I}_k \rightarrow \mathbb{C} \text{ hol.} \left| \begin{array}{l} f(e, e') = \Delta(e, e' - c')f(c, c') \\ \text{for all Peirce equivalent } (e, e'), (c, c') \end{array} \right. \right\} . \end{aligned}$$

We note that due to Lemma 3.2, $\Delta(e, e' - c') = \Delta'_e(c')$ for $\mathbf{e} = (e, e')$, which is the (opposite) Jordan algebra determinant defined by the idempotent \mathbf{e} . The main result of this section asserts, that the decomposition of $H^0(X, \mathcal{L})$ into L -types is obtained by restricting the sections to the closed L -orbits.

Theorem 5.3. *The restriction map*

$$(16) \quad \rho: H^0(X, \mathcal{L}) \rightarrow \bigoplus_{k=0}^r H^0(X_k, \mathcal{L}|_{X_k}), \quad f \mapsto (f|_{X_0}, \dots, f|_{X_r})$$

is an isomorphism of L -modules.

Proof. For convenience, we set $\mathcal{H} := H^0(X, \mathcal{L})$ and $\mathcal{H}_k := H^0(X_k, \mathcal{L}|_{X_k})$. Since X_k are closed L -orbits, each \mathcal{H}_k is an irreducible representation of L , and ρ is an L -equivariant map. To show that ρ is onto, we use an inductive argument on $k = 0, \dots, r$ for the map $\rho_k: \mathcal{H} \rightarrow \mathcal{H}_{r-k} \oplus \dots \oplus \mathcal{H}_r$. Consider the map $f(x, y) := \Delta(x, y - c')$ for $(x, y) \in V \times V'$ and some fixed $\mathbf{c} = (c, c') \in \mathcal{I}$. Applying the basic identity $\Delta(u, v)\Delta(u^v, w) = \Delta(u, w + v)$ of the Jordan pair determinant, one immediately verifies that f is an element of \mathcal{H} . The restriction of f to \mathcal{I}_k is given by $f(e, e') = \Delta'_e(c')$ with $\mathbf{e} = (e, e')$. Therefore, Lemma 3.3 implies that $f|_{\mathcal{I}_k} = 0$ if and only if $\text{rk } \mathbf{c} < k$. For $k = 0$, choose $\mathbf{c} \in \mathcal{I}$ with $\text{rk } \mathbf{c} = r$. Then, $\rho_0(f)$ is non-trivial in \mathcal{H}_r , and since \mathcal{H}_r is irreducible and ρ_0 is L -equivariant, Schur's lemma implies that ρ_0 is onto. For $k > 0$, choose $\mathbf{c} \in \mathcal{I}$ with $\text{rk } \mathbf{c} = r - k$. Then, $\rho_k(f)$ is non-trivial only in the component of \mathcal{H}_{r-k} . Therefore, the L -module generated by $\rho_k(f)$ in $\mathcal{H}_{r-k} \oplus \dots \oplus \mathcal{H}_r$ is a non-trivial submodule of the first component, and by irreducibility of \mathcal{H}_{r-k} , we obtain

$$(17) \quad \mathcal{H}_{r-k} \oplus \{0\} \oplus \dots \oplus \{0\} \subseteq \rho_{r-k}(\mathcal{H}).$$

Now let (f_{r-k}, \dots, f_r) be any element in $\mathcal{H}_{r-k} \oplus \dots \oplus \mathcal{H}_r$. By induction hypothesis, there exists an element $f \in \mathcal{H}$ with $f|_{\mathcal{I}_\ell} = f_\ell$ for all $\ell > r - k$. Due to (17), the first component can be fixed by choosing a section $g \in \mathcal{H}$ with $g|_{\mathcal{I}_{r-k}} = f_{r-k} - f|_{\mathcal{I}_{r-k}}$ and $g|_{\mathcal{I}_\ell} = 0$ for $\ell > r - k$. Therefore, $\rho_k(f + g) = (f_{r-k}, \dots, f_r)$, and hence ρ_k is surjective.

To show that ρ is injective, we have to show that a section $f \in H^0(X, \mathcal{L})$ that vanishes along all closed L -orbits $(X_k)_{k=0, \dots, r}$ must also vanish on all of X . For this we use an inductive argument showing

$$(18) \quad f(e + c, e') = 0 \quad \text{for all orthogonal } \mathbf{e} = (e, e'), \mathbf{c} = (c, c') \in \mathcal{I}.$$

From Proposition 3.4 and the fact that any element admits a completion to an idempotent, it follows that any element in $V \times V'$ is projectively equivalent to some element of the form $(e + c, e')$ as in (18). Therefore, showing (18), also proves that ρ is injective. As above, let r be the rank of (V, V') . We prove (18) by induction on $n = r - k$ with $k = \text{rk}(\mathbf{e})$. For $n = 0$, the statement just reads $f(e, e') = 0$ for all

$\mathbf{e} = (e, e') \in \mathcal{I}_r$, since $c \in V_0(\mathbf{e}) = \{0\}$. This is satisfied by the assumption $f|_{\mathcal{I}_r} = 0$. For $n > 0$, we start a second proof by induction, namely induction on $\text{rk}(\mathbf{c})$. For $\text{rk}(\mathbf{c}) = 0$, i.e., $c = 0$, we have $f(e, e') = 0$ due to the assumption that $f|_{\mathcal{I}_k} = 0$. For $\text{rk}(\mathbf{c}) \geq 1$, consider the decomposition $\mathbf{c} = \mathbf{c}_1 + \mathbf{c}_2$ for some orthogonal idempotents $\mathbf{c}_i = (c_i, c'_i)$ with $\text{rk}(\mathbf{c}_2) = 1$. Due to the cocycle condition on f , we obtain for any $t \in \mathbb{C}$

$$f(e + c_1 + tc_2, e') = \Delta(e + c_1 + tc_2, -(1 - \frac{1}{t})c'_2) f(e + c_1 + c_2, e' + (1 - \frac{1}{t})c'_2).$$

By orthogonality of \mathbf{e} , \mathbf{c}_1 and \mathbf{c}_2 , the Jordan determinant simplifies to

$$\Delta(tc_2, -(1 - \frac{1}{t})c'_2) = \Delta(c_2, (1 - t)c'_2) = \Delta'_c(tc'_2) = t^{\text{rk}(\mathbf{c}_2)} = t,$$

and we therefore obtain

$$f(e + c_1 + tc_2, e') = t g(\frac{1}{t}) \quad \text{with} \quad g(\frac{1}{t}) := f(e + c_1 + c_2, e' + (1 - \frac{1}{t})c'_2)$$

for some entire function $g \in \mathcal{O}(\mathbb{C})$. Since the left hand side is holomorphic in t , it follows that $g(\frac{1}{t}) = \alpha + \beta \frac{1}{t}$ for some $\alpha, \beta \in \mathbb{C}$. By induction hypothesis (on n), $g(0) = 0$, so $\alpha = 0$, and we conclude

$$f(e + c_1 + tc_2, e') = \beta = \text{const.}$$

Setting $t = 0$, the induction hypothesis on $\text{rk}(\mathbf{c})$ yields $\beta = f(u_+ + e_+, u_-) = 0$, and hence we also obtain $f(e + c, e') = f(e + c_2 + c_1, e') = 0$. This finally proves (18), and completes the proof of the theorem. \square

The next proposition determines the highest weights of the L -types $H^0(X_j, \mathcal{L}|_{X_j})$, and gives an explicit description of a highest weight vector. Let (e_1, \dots, e_r) be the frame of tripotents associated to the strongly orthogonal roots $\gamma_1, \dots, \gamma_r$, cf. Section 3.4, and let λ be the fundamental weight associated to γ_1 . Set $\epsilon_k := \sum_{i=1}^k e_i$ for $k = 0, \dots, r$.

Proposition 5.4. *For $k \in \{0, \dots, r\}$, the L -type $H^0(X_k, \mathcal{L}|_{X_k})$ has highest weight (with respect to Φ_L^+)*

$$\lambda_k := \lambda + \gamma_1 + \dots + \gamma_k \in (\mathfrak{it})^*.$$

In particular, the decomposition (16) is multiplicity free. Moreover, the map

$$f_k(x, y) := \Delta(x, y - \bar{\epsilon}_k)$$

defines a section $f_k \in H^0(X, \mathcal{L})$, the restriction of which to X_k is a highest weight vector of $H^0(X_k, \mathcal{L}|_{X_k})$. In local coordinates $\iota_0 : V \hookrightarrow X$, the trivialization \tilde{f}_k of f_k is given by $\tilde{f}_k(x) = \Delta(x, -\bar{\epsilon}_k)$ for $x \in V$, and the restriction to the \mathbb{C} -linear span of the frame (e_1, \dots, e_r) is

$$\tilde{f}_k(x) = (1 + x_1) \cdots (1 + x_k) \quad \text{for} \quad x = x_1 e_1 + \dots + x_r e_r$$

with $x_1, \dots, x_r \in \mathbb{C}$.

Proof. Let Q_k be the L -stabilizer of the element $[\epsilon_k : \bar{\epsilon}_k] \in X_k$. According to Proposition 5.2, Q_k is also the L -stabilizer of the Peirce 2-space $V_2(\epsilon_k)$, and Lemma 5.5 below shows that the opposite Borel subgroup B_L^{op} (i.e., the one with Lie algebra $\mathfrak{b}_L^{\text{op}} := \mathfrak{h} \oplus \mathfrak{n}_L^-$) is contained in Q_k .

Therefore, to determine the weight of $H^0(X_k, \mathcal{L}|_{X_k})$, it suffices to determine the character χ_k corresponding to the line bundle $\mathcal{L}|_{X_k}$. Due to (15), this is the same character as the one corresponding to $\iota_k^* \mathcal{L}$ on \mathcal{P}_k . Hence, χ_k can be read off from the cocycle which describes sections in $\iota_k^* \mathcal{L}$, namely

$$\chi_k : Q_k \rightarrow \mathbb{C}, \quad h \mapsto \chi(h) \cdot \Delta(\epsilon_k, \bar{\epsilon}_k - h^\# \bar{\epsilon}_k) = \chi(h) \cdot \Delta'_{\epsilon_k}(h^\# \bar{\epsilon}_k),$$

where χ is the character of \mathcal{L} (restricted to $Q_k \subseteq L$). We note that Δ'_{ϵ_k} is the (opposite) Jordan algebra determinant, and one can show that $\Delta'_{\epsilon_k}(h^\# \bar{\epsilon}_k) = \Delta_{\epsilon_k}(h^{-1} \epsilon_k)$, which is a more common description. Here, we prefer to use the first formula for χ_k . Let \mathfrak{q}_k denote the Lie algebra of Q_k . Next, we determine the derivative of χ_k at $e \in Q_k$ along an element $X \in \mathfrak{q}_k$. By definition, the derivative of the character χ of \mathcal{L} is the linear functional λ . For the second term, $\tilde{\chi}_k(h) := \Delta(\epsilon_k, \bar{\epsilon}_k - h^\# \bar{\epsilon}_k)$, one shows (using standard properties of the Jordan pair determinant) that $\frac{\partial}{\partial y} \Delta(x, y)(z) = -\frac{1}{p} \Delta(x, y) \tau(x^y, z)$, where p is defined by (7). By a short computation we thus obtain

$$d\tilde{\chi}_k(e)(X) = \frac{1}{p} \tau(\epsilon_k, X^\# \bar{\epsilon}_k) = \frac{1}{p} \tau(X \epsilon_k, \bar{\epsilon}_k).$$

Recall that the Cartan subalgebra $\mathfrak{h} \subseteq \mathfrak{q}_k$ decomposes into $\mathfrak{h} = \mathfrak{s}_{\mathbb{C}} \oplus \mathfrak{s}_{\mathbb{C}}^\perp$, where the elements $H \in \mathfrak{s}_{\mathbb{C}}^\perp$ have the property that $H(e_j) = 0$ for all j . Therefore, $d\tilde{\chi}_k(e)$ vanishes along $\mathfrak{s}_{\mathbb{C}}^\perp$. Moreover, $\mathfrak{s}_{\mathbb{C}}$ is spanned by the elements $H_{\gamma_j} = D_{e_j, \bar{e}_j}$ for $j = 1, \dots, r$, and the central element Id_V , and we obtain by (strong) orthogonality of the tripotents of the frame elements

$$d\tilde{\chi}_k(e)(H_{\gamma_j}) = \frac{1}{p} \tau(D_{e_j, \bar{e}_j} \epsilon_k, \bar{\epsilon}_k) = \begin{cases} \frac{2}{p} \tau(e_j, \bar{e}_j) = 2 & , \text{ if } j \leq k, \\ 0 & , \text{ if } j > k. \end{cases}$$

Together with $d\chi(e)(\text{Id}_V) = k$, it follows that $d\tilde{\chi}_k(e)$ coincides on \mathfrak{h} with the functional $\gamma_1 + \dots + \gamma_k$, and we therefore conclude that $d\chi_k(e) = \lambda + \gamma_1 + \dots + \gamma_k$. Next, we consider the map f_k .

Due to the identity $\Delta(u, v)\Delta(u^v, w) = \Delta(u, w + v)$ it follows that f_k satisfies the cocycle condition for sections in $H^0(X, \mathcal{L})$. The evaluation of $\tilde{f}_k(x) = f_k(x, 0) = \Delta(x, -\bar{\epsilon}_k)$ at $x = x_1 e_1 + \dots + x_r e_r$ is a special case of the formula $\Delta(x, y) = \prod_{i=1}^r (1 - x_i y_i)$ with $y = y_1 \bar{e}_1 + \dots + y_r \bar{e}_r$, cf. [Lo75, §16.15]. To prove that f_k is a highest weight vector of the L -type $H^0(X_k, \mathcal{L}|_{X_k})$ it suffices to show that f_k is constant on an open subset of X_k which contains $[\epsilon_k : \bar{\epsilon}_k]$, since $B_L^{\text{op}} \subseteq Q_k$ (due to the next lemma). Indeed, f_k is even constant on the open subset $U_{\bar{\epsilon}_k} = \{[x : \bar{\epsilon}_k] \mid x \in V\} \subseteq X$, since $f_k(x, \bar{\epsilon}_k) = \Delta(x, 0) = 1$ for all $x \in V$. This completes the proof. \square

Lemma 5.5. *Let (e_1, \dots, e_r) denote the frame of tripotents corresponding to the Cartan subalgebra \mathfrak{h} of \mathfrak{l} as it is defined in Section 2. For $k = 0, \dots, r$ let $\epsilon_k := \sum_{i=1}^k e_i$. Then, the stabilizer subgroup $Q_k \subseteq L$ of the Peirce 2-space $V_2(\epsilon_k)$ contains the opposite Borel subgroup B_L^{op} .*

Proof. This proof is based on the ideas of [Up86]. Since the Peirce variety \mathcal{P}_k is a projective variety, the stabilizer subgroup $Q_k \subseteq L$ is parabolic, and hence it suffices to prove this lemma on the Lie algebra level, i.e., we show that $\mathfrak{b}_L^{\text{op}} \subseteq \mathfrak{q}_k$ with $\mathfrak{q}_k = \{T \in \mathfrak{l} \mid TV_2(\epsilon_k) \subseteq V_2(\epsilon_k)\}$. Let $V = \bigoplus_{0 \leq i \leq j \leq r} V_{ij}$ be the joint Peirce decomposition with respect to the frame of tripotents (e_1, \dots, e_r) , set $V_{ij} := V_{ji}$ for $i \neq j$, and define

$$\mathfrak{l}_{ij} := \{D_{u,\bar{v}} \in \mathfrak{l} \mid u \in V_{i\ell}, v \in V_{\ell j} \text{ for some } \ell\}.$$

We note that $\mathfrak{l}_{ij} \neq \mathfrak{l}_{ji}$ for $i \neq j$. We claim that

$$\mathfrak{l} = \mathfrak{l}_0 \oplus \bigoplus_{i \neq j} \mathfrak{l}_{ij} \quad \text{with} \quad \mathfrak{l}_0 := \bigoplus_{i=1}^r \mathfrak{l}_{ii}$$

is the weight space decomposition of \mathfrak{l} with respect to the adjoint action of the abelian subalgebra $\mathfrak{s} = \langle i D_{e_j, \bar{e}_j}, i \text{Id}_V \mid j = 1, \dots, r \rangle_{\mathbb{R}} \subseteq \mathfrak{t}$. Indeed, since $\mathfrak{l} = [\mathfrak{p}^+, \mathfrak{p}^-]$, \mathfrak{l} is generated by all derivations of the form $D_{u,\bar{v}}$ with $u, v \in V$, and for $T = D_{u,\bar{v}} \in \mathfrak{l}_{ij}$, the Jordan identity JP15 implies

$$[D_{e_k, \bar{e}_k}, D_{u,\bar{v}}] = D_{\{e_k, \bar{e}_k, u\}, \bar{v}} - D_{u, \{\bar{e}_k, e_k, \bar{v}\}} = (\delta_{ki} - \delta_{kj}) D_{u,\bar{v}},$$

where δ_{ij} is Kronecker's delta. Moreover, $i \text{Id}_V$ acts trivially on each \mathfrak{l}_{ij} . With (8), it readily follows that the weight corresponding to \mathfrak{l}_{ij} is $(\gamma_i - \gamma_j)/2$, where we also set $\gamma_0 := 0$.

Now let $\mathfrak{l} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Phi_L} \mathfrak{l}_{\alpha}$ be the root decomposition of \mathfrak{l} with respect to the maximal torus \mathfrak{t} . By [HC56, Lemma 13], a positive root $\alpha \in \Phi_L^+$ either vanishes on \mathfrak{s} , or its restriction to \mathfrak{s} is of the form $(\gamma_i - \gamma_j)/2$ with $1 \leq j < i \leq r$ or $-\gamma_i/2$. We thus obtain that

$$\mathfrak{p} := \bigoplus_{1 \leq i \leq j \leq r} \mathfrak{l}_{ij} \oplus \bigoplus_{i=1}^r \mathfrak{l}_{i0}$$

is a subalgebra of \mathfrak{l} which contains the opposite Borel subalgebra $\mathfrak{b}_L^{\text{op}}$.

In the last step of the proof, we show that $\mathfrak{p} \subseteq \mathfrak{q}_k$. Due to the derivation property of elements $T \in \mathfrak{p}$, the relation $T\epsilon_k \in V_2(\epsilon_k)$ already implies $TV_2(\epsilon_k) \subseteq V_2(\epsilon_k)$. First assume that $T \in \mathfrak{l}_{ii}$, so $T = D_{u,\bar{v}}$ with $u, v \in V_{i\ell}$. Since $V_{i\ell}$ is contained in one of the Peirce spaces $V_m(\epsilon_k)$ with $m \in \{0, 1, 2\}$, the Peirce rules imply $D_{u,\bar{v}}\epsilon_k \in V_2(\epsilon_k)$. Now assume $T \in \mathfrak{l}_{ij}$ for $1 \leq i < j \leq r$. If $i \leq k$, then the joint Peirce rules imply $D_{u,\bar{v}}\epsilon_k \in V_{ik}$, which is a subspace of $V_2(\epsilon_k)$. If $i > k$, then $j > k$ and the joint Peirce rules imply $D_{u,\bar{v}}\epsilon_k = 0$. For the last case, $T \in \mathfrak{l}_{i0}$, i.e. $T = D_{u,\bar{v}}$ with $u \in V_{i\ell}$, $v \in V_{\ell 0}$, the joint Peirce rules yield $D_{u,\bar{v}}\epsilon_k = 0$. To sum up, this shows that $T\epsilon_k \in V_2(\epsilon_k)$ for all $T \in \mathfrak{p}$, which completes the proof. \square

6. REDUCED SPACES FOR THE K -ACTION

6.1. Moment map. In this section we explicitly determine the moment map of the K -action on the compact Hermitian symmetric space $X = U/K$. Recall from Section 4.1 that X can be realized as the coadjoint orbit $\mathcal{O}_{\lambda} \subseteq \mathfrak{u}^*$ with base point $i\lambda \in \mathfrak{u}^*$, where $-\lambda$ is the extension

of the fundamental weight associated to γ_1 . In this realization the moment map $\mu : \mathcal{O}_\lambda \rightarrow \mathfrak{u}^*$ is just the restriction of the identity map to \mathcal{O}_λ , and the moment map $\mu_{\mathfrak{k}}$ corresponding to the K -action on $X \cong \mathcal{O}_\lambda$ is given by restriction to \mathfrak{k} , i.e., $\mu_{\mathfrak{k}}(x) = \mu(x)|_{\mathfrak{k}}$ for all $x \in \mathcal{O}_\lambda$.

In the following we use a constant multiple of the Killing form κ on \mathfrak{u} to identify \mathfrak{u}^* with \mathfrak{u} , \mathfrak{k}^* with \mathfrak{k} , and coadjoint orbits with the corresponding adjoint orbits. More precisely, we use

$$(19) \quad \vartheta : \mathfrak{u} \rightarrow \mathfrak{u}^*, \quad X \mapsto -\frac{1}{2p} \kappa(X, \square),$$

where p is defined in (7). Let $\tilde{\mu} := \vartheta \circ \mu$ and $\tilde{\mu}_{\mathfrak{k}} := \vartheta \circ \mu_{\mathfrak{k}}$ denote the corresponding moment maps. Then, $\tilde{\mu}_{\mathfrak{k}} = \text{pr}_{\mathfrak{k}} \circ \tilde{\mu}$, where $\text{pr}_{\mathfrak{k}}$ is the orthogonal projection of \mathfrak{u} onto \mathfrak{k} with respect to the Killing form.

Lemma 6.1. *The map ϑ identifies $i\lambda \in \mathfrak{u}^*$ with $i\text{Id}_V \in \mathfrak{k}$, which is central in \mathfrak{k} .*

Proof. Let $\eta \in \mathfrak{u}$ be the unique element determined by the relation $i\lambda(Y) = \kappa(X, Y)$ for all $Y \in \mathfrak{u}$. We have to show that $X = -\frac{i}{2p} \text{Id}_V$. Since λ vanishes on \mathfrak{k}^\perp , it follows that $X \in \mathfrak{k}$. In addition, the relation $(\text{Ad}_u^* \lambda)(Y) = \kappa(X, \text{Ad}_u^{-1} Y) = \kappa(\text{Ad}_u X, Y)$ implies that the stabilizer of the coadjoint action of U with respect to λ (which is K) coincides with the stabilizer of the adjoint action of U with respect to X . Therefore, X is central in \mathfrak{k} , and since $\mathfrak{z}(\mathfrak{k}) = \mathbb{R} \cdot i\text{Id}_V$, we conclude that $X = c \cdot i\text{Id}_V$ for some $c \in \mathbb{R}$. Finally, using (6), the relation $\lambda(H_{\gamma_1}) = -1$ with $H_{\gamma_1} = D_{e_1, \bar{e}_1}$ implies

$$-i = (i\lambda)(H_{\gamma_1}) = \kappa(c \cdot i\text{Id}_V, D_{e_1, \bar{e}_1}) = 2i c \cdot \text{Tr } D_{e_1, \bar{e}_1} = 2i c \cdot \tau(e_1, \bar{e}_1),$$

and since $p = \tau(e_1, \bar{e}_1)$, this completes the proof. \square

As a first step, we determine the moment map on the open and dense subset $V \subseteq X$. Recall the Jordan-theoretic description of X via projective equivalence, $X \cong (V \times V')/R$, so elements of X are equivalence classes $[x : a]$ of elements in $V \times V'$. The embedding $\iota_0 : V \hookrightarrow X$ is given by $\iota_0(x) = [x : 0]$. The moment map on $X = U/K$ is by definition identified with the moment map $\tilde{\mu}_{\mathfrak{k}}$ on $\vartheta^{-1}(\mathcal{O}_\lambda) \subseteq \mathfrak{u}$ via the isomorphism given by $uK \mapsto \text{Ad}_u(i\text{Id}_V)$.

Proposition 6.2. *The restriction of the moment map $\tilde{\mu}_{\mathfrak{k}} : X \rightarrow \mathfrak{k}$ of the K -action on X to $V \subseteq X$ is given by*

$$(20) \quad \tilde{\mu}_{\mathfrak{k}} : V \rightarrow \mathfrak{k}, \quad x \mapsto i \left(B_{x, -\bar{x}}^{-1} - Q_x B_{-\bar{x}, x}^{-1} Q_{\bar{x}} \right).$$

If $x = \sum_{j=1}^k \sigma_j e_j$ is the spectral decomposition of $x \in V$, then

$$(21) \quad \tilde{\mu}_{\mathfrak{k}}(x) = i \left(\text{Id}_V - \sum_{j=1}^k \frac{\sigma_j^2}{1 + \sigma_j^2} D_{e_j, \bar{e}_j} \right).$$

Proof. Let $\pi : U \rightarrow X \cong U/K$ be the canonical projection of U onto X given by $u \mapsto u[0 : 0]$, then the embedding $\iota_0 : V \hookrightarrow X$ of V into X admits a lift to U given by

$$\varphi : V \rightarrow U, \quad x \mapsto u_x := t_x \circ B_{x, -\bar{x}}^{1/2} \circ \tilde{t}_{\bar{x}},$$

Indeed, since $\tilde{t}_{\bar{x}}[0 : 0] = [0 : \bar{x}] = [0 : 0]$, we obtain

$$\pi \circ \varphi(x) = u_x[0 : 0] = t_x \circ B_{x, -\bar{x}}^{1/2}[0 : 0] = t_x[0 : 0] = [x : 0] = \iota_0(x).$$

Therefore, the moment map on $V \subseteq X$ is given by $\tilde{\mu} : V \rightarrow \mathfrak{u}$ with $\tilde{\mu}^x := \tilde{\mu}(x) = \text{Ad}_{u_x}(i \text{Id}_V)$. Recall that all Lie algebras are realized as vector fields on V , and that the adjoint action reads

$$(\text{Ad}_{u^{-1}} \zeta)(z) = du(z)^{-1} \cdot \zeta(u(z)).$$

Explicitly, since $u_x^{-1}(z) = t_{-x} B_{x, -\bar{x}}^{1/2} \tilde{t}_{-\bar{x}}(z) = B_{x, -\bar{x}}^{1/2} z^{-\bar{x}} - x$, we obtain

$$\begin{aligned} \tilde{\mu}^x(z) &= i \left(B_{x, -\bar{x}}^{1/2} B_{z, -\bar{x}}^{-1} \right)^{-1} \cdot (i u_x^{-1}(z)) \\ &= i B_{z, -\bar{x}} z^{-\bar{x}} - i B_{z, -\bar{x}} B_{x, -\bar{x}}^{-1/2} x \\ &= i \left(z + Q_z \bar{x} - B_{z, -\bar{x}}(x^{-\bar{x}}) \right). \end{aligned}$$

In the last step we used the identity $B_{x, -\bar{x}}^{-1/2} x = x^{-\bar{x}}$, which is obvious when x is replaced by its spectral decomposition $x = \sum \sigma_j e_j$. Recall that the moment map $\tilde{\mu}_{\mathfrak{k}}$ of the K -action on $V \subseteq X$ is given by the orthogonal projection of $\tilde{\mu}^x$ onto \mathfrak{k} . The orthogonal projection yields the linear terms of the vector field $\tilde{\mu}^x$, therefore,

$$\tilde{\mu}_{\mathfrak{k}}^x(z) = d\tilde{\mu}^x(0) \cdot z = i(z - D_{z, \bar{x}}(x^{-\bar{x}})) = i(\text{Id}_V - D_{x^{-\bar{x}}, \bar{x}})(z).$$

From this, formula (20) follows with a short calculation by using the relations $B_{x, y} D_{x^y, z} = D_{x, z} - Q_x Q_{y, z}$ (JP30) and $B_{x, y} Q_{x^y} = Q_{x^y} B_{y, x} = Q_x$ (JP28), and formula (21) follows by using the spectral decomposition $x = \sum \sigma_j e_j$. This completes the proof. \square

For the extension of the result of Proposition 6.2 to all of X , we need the following operator.

Lemma 6.3. *For $(x, a) \in (V, V')$, let $\Gamma_{x, a} := B_{x, a} B_{x^a, -\bar{x}^a} B_{\bar{a}, \bar{x}}$. Then*

- (a) $\Gamma_{x, a}$ depends polynomially on x, \bar{x}, a, \bar{a} .
- (b) If $[x : a] = [z : b]$, then $\Gamma_{z, b} = B_{x, a-b}^{-1} \Gamma_{x, a} B_{\bar{a}-\bar{b}, \bar{x}}^{-1}$.
- (c) $\Gamma_{x, a}$ is a positive definite operator on V with respect to the inner product $(-|-)$ defined in (3).

Mutatis mutandis, the same results hold for the adjoint operator $\Gamma_{\bar{x}, \bar{a}} := \Gamma_{x, a}^\# = B_{\bar{x}, \bar{a}} B_{\bar{x}^a, -x^a} B_{a, x} \in \text{End}(V')$ where the adjoint is taken with respect to the trace form τ .

Proof. The proof of (a) and (b) relies on the fact that the Bergman operator satisfies the relations $B_{u, v} B_{u^v, w} = B_{u, v+w}$ and $B_{w, u^v} B_{v, u} =$

$B_{v+w,u}$. For (a), consider the operator $\Gamma'_{x,a,y,b} := B_{x,a}B_{x^a,-y^b}B_{b,y}$ with $(x,a) \in (V,V')$ and $(y,b) \in (V',V)$. Then

$$\Gamma'_{x,a,y,b} = B_{x,-y^b+a}B_{b,y} = B_{x,a}B_{-x^a+b,y^b},$$

and the first of these identities implies that $\Gamma'_{x,a,y,b}$ is polynomial in $(x,a) \in V$, whereas the second identity implies that $\Gamma'_{x,a,y,b}$ is polynomial in $(y,b) \in (V',V)$. Therefore, $\Gamma_{x,a} = \Gamma'_{x,a,\bar{x},\bar{a}}$ is polynomial in x,a,\bar{x},\bar{a} . Part (b) is a simple application of the identities satisfied by the Bergman operators, since $[z:b] = [x:a]$ implies $z = x^{a-b}$. For (c), we first note that $\Gamma_{x,a}$ is self-adjoint with respect to $(-|-)$. Due to (b) and Proposition 3.4, it suffices to prove (c) for $(x,a) = (e+z,\bar{e})$ for some tripotent $e \in S$ and $z \in V_0(e)$. We calculate $\Gamma_{e+z,\bar{e}}$ by means of the limit $\lim_{t \rightarrow 1} \Gamma_{e+z,t\bar{e}}$ with $t \in \mathbb{R} \setminus \{1\}$. Using the relation $(e+z)^{t\bar{e}} = \frac{1}{1-t} e + z$ and the Peirce rules, it is straightforward to obtain

$$(22) \quad \Gamma_{e+z,\bar{e}} = \lim_{t \rightarrow 1} \Gamma_{e+z,t\bar{e}} = \lim_{t \rightarrow 1} B_{(1-t)e, -(1-t)\bar{e}} B_{z,-\bar{z}} = B_{z,-\bar{z}}.$$

Let $z = \sum_{i=1}^k \sigma_i e_i$ be the spectral decomposition of z , and let $V = \bigoplus_{0 \leq i \leq j \leq k} V_{ij}$ be the joint Peirce decomposition with respect to the orthogonal system of tripotents (e_1, \dots, e_k) . Then [Lo77, §3.15], the relation $B_{z,-\bar{z}} v_{ij} = (1 + \sigma_i^2)(1 + \sigma_j^2) v_{ij}$ holds for all $v_{ij} \in V_{ij}$, where $\lambda_0 := 0$. This shows that $B_{z,-\bar{z}}$ and hence $\Gamma_{e+z,\bar{e}}$ is a positive definite operator on V . \square

Theorem 6.4. *The moment map $\tilde{\mu}_{\mathfrak{k}} : X \rightarrow \mathfrak{k}$ of the K -action on X is given by*

$$(23) \quad \tilde{\mu}_{\mathfrak{k}}([x:a]) = i \left(B_{\bar{a},\bar{x}} \Gamma_{x,a}^{-1} B_{x,a} - Q_x \Gamma_{\bar{x},\bar{a}}^{-1} Q_{\bar{x}} \right)$$

If $(e+z,\bar{e}) \in [x:a]$ is a representative as in Proposition 3.4, and $z = \sum_{j=1}^k \sigma_j e_j$ is the spectral decomposition of z , then

$$(24) \quad \tilde{\mu}_{\mathfrak{k}}([x:a]) = i \left(\text{Id}_V - \sum_{j=1}^k \frac{\sigma_j^2}{1+\sigma_j^2} D_{e_j,\bar{e}_j} - D_{e,\bar{e}} \right).$$

Proof. For (23), it suffices to check that $\tilde{\mu}_{\mathfrak{k}}([x:a])$ is well-defined and that its restriction to $V \subseteq X$ coincides with (20). The latter part is easy to see, since for $a = 0$, we obtain $B_{x,0} = \text{Id}_V$ and $\Gamma_{x,0} = B_{x,-\bar{x}}$. To prove that $\tilde{\mu}_{\mathfrak{k}}([x:a])$ is well-defined, let $[x:a] = [z:b]$, so $z = x^{a-b}$. Due to Lemma 6.3, $\Gamma_{z,b} = B_{x,a-b}^{-1} \Gamma_{x,a} B_{\bar{a}-b,\bar{x}}^{-1}$, and hence

$$\begin{aligned} \tilde{\mu}_{\mathfrak{k}}([z:b]) &= i \left(B_{\bar{b},\bar{z}} \Gamma_{z,b}^{-1} B_{z,b} - Q_z \Gamma_{\bar{z},\bar{b}}^{-1} Q_{\bar{z}} \right) \\ &= i \left(B_{\bar{b},\bar{x}^{\bar{a}-\bar{b}}} B_{\bar{a}-b,\bar{x}}^{-1} \Gamma_{x,a}^{-1} B_{x,a-b} B_{x^{a-b},b} + Q_{x^{a-b}} B_{a-b,x} \Gamma_{\bar{x},\bar{a}}^{-1} B_{\bar{x},\bar{a}-b} Q_{\bar{x}^{\bar{a}-\bar{b}}} \right) \\ &= i \left(B_{\bar{a},\bar{x}} \Gamma_{x,a}^{-1} B_{x,a} - Q_x \Gamma_{\bar{x},\bar{a}}^{-1} Q_{\bar{x}} \right) \\ &= \tilde{\mu}_{\mathfrak{k}}([x:a]). \end{aligned}$$

We note that continuity also yields the condition $\tilde{\mu}_{\mathfrak{k}}([x:a]) \in \mathfrak{k}$. The identity (24) follows by continuity from (21), since $[x:a] = [e+z:\bar{e}] =$

$\lim_{t \rightarrow 1} [\frac{1}{1-t}e + z : 0]$, and the spectral decomposition of $\frac{1}{1-t}e + z$ is just $\frac{1}{1-t}e + z = \frac{1}{1-t}e + \sum \sigma_j e_j$, since $z \in V_0(e)$. \square

Corollary 6.5. *The image of the moment map $\tilde{\mu}_{\mathfrak{k}}$ in \mathfrak{k} is given by*

$$\tilde{\mu}_{\mathfrak{k}}(X) = \left\{ i \left(\text{Id}_V - \sum_{i=1}^r \nu_i D_{e_i, \bar{e}_i} \right) \left| \begin{array}{l} (e_1, \dots, e_r) \text{ frame of tripotents,} \\ (\nu_1, \dots, \nu_r) \in [0, 1]^r \end{array} \right. \right\}.$$

In the last part of this section, we return to the moment map $\mu_{\mathfrak{k}}$ in its original form with image in \mathfrak{k}^* , and determine the intersection of $\mu_{\mathfrak{k}}(X)$ with the closed positive Weyl chamber $i\mathfrak{c} \subseteq \mathfrak{k}^*$ (with respect to Φ_L^+), i.e.,

$$\mathfrak{c} := \{ \alpha \in (i\mathfrak{t})^* \mid \alpha(H_\beta) \geq 0 \text{ for all } \beta \in \Phi_L^+ \}.$$

Recall from Section 4.1 that real valued functionals $\alpha \in (i\mathfrak{t})^*$ are identified with their complex extensions to functionals on \mathfrak{l} (zero-extension on the orthogonal complement of $i\mathfrak{t}$), and the restriction of $i\alpha$ to \mathfrak{k} is a real valued functional on \mathfrak{k} , so $i\mathfrak{c} \subseteq \mathfrak{k}^*$.

Theorem 6.6. *The intersection of the image of the moment map $\mu_{\mathfrak{k}}$ with $i\mathfrak{c}$, is given by*

$$\mu_{\mathfrak{k}}(X) \cap i\mathfrak{c} = \left\{ i \left(\lambda + \sum_{j=1}^r \nu_j \gamma_j \right) \in \mathfrak{k}^* \left| 1 \geq \nu_1 \geq \dots \geq \nu_r \geq 0 \right. \right\} =: \Pi_{\mathfrak{s}}.$$

In particular, this is a convex polytope.

Proof. Let (e_1, \dots, e_r) be the frame of tripotents that is associated to the maximal abelian subalgebra $\mathfrak{t} \subseteq \mathfrak{u}$ as described in Section 3.4. Consider

$$\tilde{\Pi}_{\mathfrak{s}} := \left\{ i \left(\text{Id}_V - \sum_{i=1}^r \nu_i D_{e_i, \bar{e}_i} \right) \left| 1 \geq \nu_1 \geq \dots \geq \nu_r \geq 0 \right. \right\} \subseteq \mathfrak{k}.$$

We first show that $\vartheta(\tilde{\Pi}_{\mathfrak{s}}) = \Pi_{\mathfrak{s}}$ with ϑ as in (19). Indeed, Lemma 6.1 shows that $\vartheta(i\text{Id}) = i\lambda$, and using (8), we obtain for all $T \in \mathfrak{k}$,

$$\begin{aligned} \vartheta(iD_{e_j, \bar{e}_j})(T) &= -\frac{i}{2p} \kappa(D_{e_j, \bar{e}_j}, T) = \frac{i}{2p} \kappa(T, [e_j, q_{\bar{e}_j}]) \\ &= \frac{i}{2p} \kappa(Te_j, q_{\bar{e}_j}) = -\frac{i}{p} \tau(Te_j, \bar{e}_j) = -i\gamma_j(T). \end{aligned}$$

Therefore, $\vartheta(iD_{e_j, \bar{e}_j}) = -i\gamma_j$, and we conclude that $\vartheta(\tilde{\Pi}_{\mathfrak{s}}) = \Pi_{\mathfrak{s}}$. Since K acts transitively on the set of frames of tripotents, Corollary 6.5 implies that $\text{Ad}_K(\tilde{\Pi}_{\mathfrak{s}}) = \tilde{\mu}_{\mathfrak{k}}(X)$, and since ϑ is K -equivariant, this also yields

$$(25) \quad \text{Ad}_K^*(\Pi_{\mathfrak{s}}) = \mu_{\mathfrak{k}}(X).$$

As a second step, we prove that $\Pi_{\mathfrak{s}} \subseteq i\mathfrak{c}$: Theorem 5.3 and Proposition 5.4 imply that for all $\ell \in \{0, 1, \dots, r\}$ the functional $\lambda_\ell := \lambda + \sum_{j=1}^\ell \gamma_j \in (i\mathfrak{t})^*$ is a highest weight of an L -type in $H^0(X, \mathcal{L})$, hence $i\lambda_\ell \in \Pi_{\mathfrak{s}} \cap i\mathfrak{c}$. Since $i\mathfrak{c}$ is convex, we conclude that

$$(26) \quad \Pi_{\mathfrak{s}} = \text{conv}\{i\lambda_0, \dots, i\lambda_r\} \subseteq i\mathfrak{c}.$$

Finally, since the closed Weyl chamber $i\mathfrak{c}$ is a fundamental domain for the K -action on \mathfrak{k}^* (cf. [DK00, Lemma 3.8.2]), it follows from (25) and (26) that $\mu_{\mathfrak{k}}(X) \cap i\mathfrak{c} = \Pi_{\mathfrak{s}}$. \square

6.2. Reduced spaces. The next goal is to show that the reduced spaces are points, so we first determine the fibre $\tilde{\mu}_{\mathfrak{k}}^{-1}(T)$ of a given element $T = i(\text{Id}_V - \sum_{j=1}^k \nu_j D_{e_j, \bar{e}_j}) \in \mathfrak{k}$.

Lemma 6.7. *Let (e_1, \dots, e_k) and (c_1, \dots, c_ℓ) be two systems of orthogonal tripotents, and let $\nu_1 < \nu_2 < \dots < \nu_k$ and $\mu_1 < \mu_2 < \dots < \mu_\ell$ be non-zero real numbers. Then*

$$\sum_{i=1}^k \nu_i D_{e_i, \bar{e}_i} = \sum_{j=1}^{\ell} \mu_j D_{c_j, \bar{c}_j} \iff k = \ell \text{ and } \nu_i = \mu_i, e_i \approx c_i \text{ for all } i.$$

Here, $e_i \approx c_i$ means that e_i and c_i induce the same Peirce decompositions, i.e., $V_m(e_i) = V_m(c_i)$ for $m \in \{0, 1, 2\}$.

Proof. For convenience, we set $A := \sum_{i=1}^k \nu_i D_{e_i, \bar{e}_i}$ and $B := \sum_{j=1}^{\ell} \mu_j D_{c_j, \bar{c}_j}$. Let $V = \bigoplus_{0 \leq i \leq j \leq r} V_{ij}$ be the joint Peirce decomposition with respect to the orthogonal family (e_1, \dots, e_k) . Then, the Peirce rules imply $Ax_{ij} = (\nu_i + \nu_j)x_{ij}$ for $x_{ij} \in V_{ij}$, where we also set $\nu_0 := 0$. Therefore, V decomposes into eigenspaces of A , and the eigenspace of the highest eigenvalue, namely $2\nu_k$, is $V_2(e_k)$. In the same way we obtain a decomposition of V into eigenspaces of B , and the eigenspace of the highest eigenvalue $2\mu_\ell$ is $V_2(c_\ell)$. Assuming $A = B$, we therefore obtain $\nu_k = \mu_\ell$ and $V_2(e_k) = V_2(c_\ell)$. Since Peirce 2-spaces corresponding to tripotents uniquely determine the whole Peirce decomposition, this also implies $V_m(e_k) = V_m(c_\ell)$ for $m \in \{0, 1, 2\}$, and hence $D_{e_k, \bar{e}_k} = D_{c_\ell, \bar{c}_\ell}$. Therefore, the assumption $A = B$ is reduced to $\sum_{i=1}^{k-1} \nu_i D_{e_i, \bar{e}_i} = \sum_{j=1}^{\ell-1} \mu_j D_{c_j, \bar{c}_j}$ and the statement follows by induction. \square

Theorem 6.8 (Reduced spaces). *Let $\tilde{\mu}_{\mathfrak{k}} : X \rightarrow \mathfrak{k}$ be the moment map of the K -action on X , and let $T \in \tilde{\mu}_{\mathfrak{k}}(X)$, i.e., $T = \tilde{\mu}_{\mathfrak{k}}([e_0 + z : \bar{e}_0])$ for some tripotent $e_0 \in S$ and $z \in V_0(e_0)$. Let $z = \sum_{j=1}^k \sigma_j e_j$ be the spectral decomposition of z .*

(a) *The fibre of T with respect to the moment map is given by*

$$\tilde{\mu}_{\mathfrak{k}}^{-1}(T) = \left\{ \left[c_0 + \sum_{j=1}^k \sigma_j c_j : \bar{c}_0 \right] \middle| \begin{array}{l} c_j \in S, c_j \approx e_j \\ \text{for all } j = 0, \dots, k \end{array} \right\}.$$

(b) *The stabilizer subgroup K_T of T consists of those elements leaving the Peirce spaces of the joint Peirce decomposition $V = \bigoplus V_{ij}$ with respect to the orthogonal system (e_1, \dots, e_k, e) invariant.*

(c) *The reduced space $\tilde{\mu}_{\mathfrak{k}}^{-1}(T)/K_T$ is a point.*

Proof. Recall from Proposition 3.4 that any element of X can be represented as $[c_0 + w : \bar{c}_0]$ for some tripotent c_0 and $w \in V_0(c_0)$. Let $w = \sum_{i=1}^{\ell} \tau_i c_i$ be the spectral decomposition of w . Now assume that

$\tilde{\mu}_{\mathfrak{k}}([e_0 + z : \bar{e}_0]) = \tilde{\mu}_{\mathfrak{k}}([c_0 + w : \bar{c}_0])$. Then, Theorem 6.4 and Lemma 6.7 imply that $k = \ell$ and $\sigma_j^2/(1 + \sigma_j^2) = \tau_j^2/(1 + \tau_j^2)$, $c_j \approx e_j$ for all $j = 0, \dots, k$. Moreover, since the identity between σ_j and τ_j is solved only for $\sigma_j = \pm \tau_j$, and σ_j, τ_j are assumed to be positive, this proves (a). For (b), we note that $\text{Ad}_k D_{e_i, \bar{e}_i} = D_{ke_j, \overline{ke_j}}$, and hence Lemma 6.7 implies that $k \in K$ stabilizes T if and only if $e_j \approx ke_j$ for all j . Since the Peirce spaces of the joint Peirce decomposition corresponding to (e_0, e_1, \dots, e_k) can be described by intersections of the Peirce spaces $V_m(e_j)$ with $m \in \{0, 1, 2\}$, $j = 0, \dots, k$, and, conversely, the Peirce spaces $V_m(e_j)$ are given by direct sums of joint Peirce spaces. This proves (b). For (c), we have to show that K_T acts transitively on $\tilde{\mu}_{\mathfrak{k}}^{-1}(T)$. Due to (a), this is equivalent to the statement that K_T conjugates any orthogonal systems (e_0, \dots, e_k) and (c_0, \dots, c_k) of tripotents with $e_j \approx c_j$ for all j . This follows from the fact [Lo77, §5.9] that K acts transitively on the set of frames of tripotents: since each e_j and c_j can be decomposed further into orthogonal primitive tripotents to obtain frames, and since $\text{rk } e_j = \text{rk } c_j$, there exists an element $k \in K$ mapping c_j onto e_j for all j . Since $c_j \approx e_j$, k preserves the Peirce spaces $V_m(e_j) = V_m(c_j)$. By the same argument as for (b), it follows that $k \in K_T$. \square

7. BRANCHING LAWS

Before we turn to the problem of decomposing the spaces $H^0(X, \mathcal{L}^k)$ under the group K we recall some notions from the theory of Hamiltonian actions of compact Lie groups. For a more thorough treatment, we refer to [GS82] and [S95].

Let K temporarily denote an arbitrary compact connected Lie group which acts holomorphically on the connected compact Kähler manifold (M, Ω) in a Hamiltonian fashion, and let $\tau : M \rightarrow \mathfrak{k}^*$ be the moment map for the action. Assume that $\mathcal{L} \rightarrow M$ is a prequantum line bundle for (M, Ω) . Then K acts holomorphically on \mathcal{L} as bundle isomorphisms, and this action extends to an action of the universal complexification, $K^{\mathbb{C}}$, of K . We recall two notions of stability for the K -action on M . Firstly, we have the set

$$M_{ss}(\mathcal{L}) := \{m \in M \mid s(m) \neq 0 \text{ for some } k \in \mathbb{N}, \text{ and } s \in H^0(M, \mathcal{L}^k)^K\}$$

of *algebraically semistable* points. Secondly, we have the set

$$M_{ss} = \{m \in M \mid \overline{K^{\mathbb{C}} \cdot m} \cap \tau^{-1}(0) \neq \emptyset\}$$

of *analytically semistable* points. In fact, the identity

$$M_{ss}(\mathcal{L}) = M_{ss}$$

holds (cf. [S95, Thm. 2.18]).

We now turn to our particular setting, where K again is the stabilizer in U of $eQ \in X$. Let \mathcal{O}_{ξ}^K be a coadjoint orbit in \mathfrak{k}^* through an integral $i\xi \in \mathfrak{t}^* \subseteq \mathfrak{k}^*$, which we view as a symplectic manifold when equipped

with the Kostant-Kirillov symplectic form ω_ξ^K . Let $\overline{\mathcal{O}_\xi^K}$ denote \mathcal{O}_ξ^K equipped with the symplectic form $-\omega_\xi^K$, and with the reverse complex structure, i.e., the sheaf of holomorphic functions on $\overline{\mathcal{O}_\xi^K}$ is the sheaf of antiholomorphic functions on \mathcal{O}_ξ^K . If \mathcal{L}_ξ is the prequantum line bundle for \mathcal{O}_ξ^K , let $\overline{\mathcal{L}_\xi^*}$ denote the line bundle over $\overline{\mathcal{O}_\xi^K}$ where the fibre over x is given by the antilinear functionals $(\mathcal{L}_\xi)_x \rightarrow \mathbb{C}$. Then $\overline{\mathcal{L}_\xi^*}$ is a holomorphic prequantum line bundle for $(\overline{\mathcal{O}_\xi^K}, -\omega_\xi^K)$.

Consider now the product space $X \times \overline{\mathcal{O}_\xi^K}$. Let p_1 and p_2 denote the projections onto X and $\overline{\mathcal{O}_\xi^K}$, respectively. For $k \in \mathbb{N}$, we equip $X \times \overline{\mathcal{O}_\xi^K}$ with the symplectic form $p_1^*(k\omega_\lambda) - p_2^*\omega_\xi^K$. The diagonal action of K on $X \times \overline{\mathcal{O}_\xi^K}$, when equipped with this symplectic form, is then holomorphic and Hamiltonian with moment map

$$\mu_\mathfrak{k}^{k,\xi}(x, y) := \mu_\mathfrak{k}^k(x) - y, \quad (x, y) \in X \times \overline{\mathcal{O}_\xi^K}.$$

Put

$$\mathcal{L}(\xi, k) := p_1^*\mathcal{L}^k \otimes p_2^*\overline{\mathcal{L}_\xi^*}.$$

Then $\mathcal{L}(\xi, k)$ is a holomorphic prequantum line bundle for $((X \times \overline{\mathcal{O}_\xi^K}), p_1^*(k\omega_\lambda) - p_2^*\omega_\xi^K)$, and the K -action lifts to a holomorphic action on $\mathcal{L}(\xi, k)$.

From Theorem 6.8 (c) we immediately conclude the following result.

Proposition 7.1. *For every $k \in \mathbb{N}$, the group K acts transitively on $(\mu_\mathfrak{k}^{k,\xi})^{-1}(0) \subseteq X \times \overline{\mathcal{O}_\xi^K}$.*

Proposition 7.2. *For every $\xi \in (\mathfrak{it})^*$, and $k \in \mathbb{N}$, the dimension of the space $H^0(X \times \overline{\mathcal{O}_\xi^K}, \mathcal{L}(\xi, k))^K$ is at most one.*

Proof. First of all, by [S95, Thm. 2.18],

$$H^0(X \times \overline{\mathcal{O}_\xi^K}, \mathcal{L}(\xi, k))^K \cong H^0((X \times \overline{\mathcal{O}_\xi^K})_{ss}, \mathcal{L}(\xi, k))^K.$$

By the definition of $(X \times \overline{\mathcal{O}_\xi^K})_{ss}$ any K -invariant section, being also $K^\mathbb{C}$ -invariant, is uniquely determined by its values on $(\mu_\mathfrak{k}^{k,\xi})^{-1}(0)$. By Proposition 7.1, such a section is in fact determined by its value at some given point in $(\mu_\mathfrak{k}^{k,\xi})^{-1}(0)$. This finishes the proof. \square

Lemma 7.3. *The integral points in $\mu_\mathfrak{k}^k(X) \cap \mathfrak{ic}$ are precisely the points $k\lambda + m_1\gamma_1 + \dots + m_r\gamma_r$, where $k \geq m_1 \geq \dots \geq m_r \geq 0$.*

Proof. Clearly, every point $k\lambda + m_1\gamma_1 + \dots + m_r\gamma_r$ with $k \geq m_1 \geq \dots \geq m_r \geq 0$ is integral. For the converse inclusion, let $\xi \in \mu_\mathfrak{k}^k(X) \cap \mathfrak{ic}$. Then $\xi = k\lambda + x_1\gamma_1 + \dots + x_r\gamma_r$ for some $x_i \in \mathbb{R}$ satisfying $k \geq x_1 \geq \dots \geq x_r \geq 0$. If ξ is integral, then $x_1\gamma_1 + \dots + x_r\gamma_r$ is also integral. From the argument at the end of the proof of Lemma 2 in [Sc69] it then follows that $x_i \in \mathbb{Z}$ for $i = 1, \dots, r$. This finishes the proof. \square

Theorem 7.4. *The space $H^0(X, \mathcal{L}^k)$ decomposes under L as*

$$H^0(X, \mathcal{L}) = \bigoplus_{k \geq m_1 \geq \dots \geq m_r \geq 0} W_{(k\lambda, \mathbf{m})},$$

where $W_{(k\lambda, \mathbf{m})}$ is the irreducible L -representation with highest weight $k\lambda + \sum_{i=1}^r m_i \gamma_i$.

Proof. By Theorem 1.1 and Lemma 7.3, only representations of the form $W_{(k\lambda, \mathbf{m})}$ can occur in $H^0(X, \mathcal{L})$, and by Proposition 7.2 they can at most have multiplicity one. Thus, it suffices to prove that every such representation actually does occur. For this, we note that every $\mathbf{m} \in \mathbb{N}_0^r$ satisfying the condition $k \geq m_1 \geq \dots \geq m_r \geq 0$ can be written uniquely as $\mathbf{m} = \sum_{j=1}^k \mathbf{m}(j)$, with $\mathbf{m}(j) = (m_1(j), \dots, m_r(j)) \in \mathbb{N}_0^r$ satisfying $1 \geq m_1(j) \geq \dots \geq m_r(j) \geq 0$. By Theorem 5.3 and Proposition 5.4, for each such $\mathbf{m}(j)$ the irreducible L -representation $W_{(\lambda, \mathbf{m}(j))}$ occurs in $H^0(X, \mathcal{L})$. Let $s_j \in H^0(X, \mathcal{L})$ be an L -highest weight vector for the representation $W_{(\lambda, \mathbf{m}(j))}$. Then $s_1 \cdots s_k \in H^0(X, \mathcal{L}^k)$ is an L -highest weight vector of weight $k\lambda + \sum_{i=1}^r m_i \gamma_i$. Hence, the representation $W_{(k\lambda, \mathbf{m})}$ occurs in $H^0(X, \mathcal{L}^k)$. \square

8. THE OKOUNKOV BODY

In this section we return to the Okounkov body $\Delta(\mathcal{L}, N_L^+, v)$. The main result is an identification of $\Delta(\mathcal{L}, N_L^+, v)$ with the moment polytope $\Pi_{\mathfrak{g}}$. We also prove that the semigroup $S(\mathcal{L}, N_L^+, v)$ is finitely generated.

Let $\mathbf{m} \in \mathbb{N}_0^r$ satisfy $1 \geq m_1 \geq \dots \geq m_r \geq 0$, and let $s_{\mathbf{m}} \in H^0(X, \mathcal{L})$ be an L -highest weight vector with highest weight $\lambda + m_1 \gamma_1 + \dots + m_r \gamma_r$.

Proposition 8.1. *The semigroup $S(\mathcal{L}, N_L^+, v)$ is generated by the elements $(1, v(s_{\mathbf{m}}))$ with $1 \geq m_1 \geq \dots \geq m_r \geq 0$.*

Proof. Let $s \in H^0(X, \mathcal{L}^k)$ be an N_L^+ -invariant vector. Then s can be written as a linear combination $s = \sum_i s_i$ of L -highest weight vectors in $H^0(X, \mathcal{L}^k)$ corresponding to distinct highest weights. These weights are distinct also as \mathfrak{h} -weights. Hence, by Proposition 4.1 $v(s_i) \neq v(s_j)$ for $i \neq j$. It follows immediately from the definition of v that $v(s) = v(s_i)$ for some s_i , namely the s_i with smallest $v(s_i)$. We can thus without loss of generality assume that s is an L -highest weight vector. By Theorem 7.4, the weight of s can then be written as $k\lambda + \sum_{i=1}^r m_i \gamma_i$ with $k \geq m_1 \geq \dots \geq m_r \geq 0$. As in the proof of Theorem 7.4, $\mathbf{m} := (m_1, \dots, m_r)$ can be written uniquely as $\mathbf{m} = \sum_{j=1}^k \mathbf{m}(j)$, with $\mathbf{m}(j) = (m_1(j), \dots, m_r(j)) \in \mathbb{N}_0^r$ satisfying $1 \geq m_1(j) \geq \dots \geq m_r(j) \geq 0$. Let $s_j \in H^0(X, \mathcal{L})$ be an L -highest weight vector with weight $\lambda + \sum_{i=1}^r m_i(j) \gamma_i$. Then $s_1 \cdots s_k \in H^0(X, \mathcal{L}^k)$ is an L -highest weight vector of weight $k\lambda + \sum_{i=1}^r m_i \gamma_i$. Since the decomposition under L is multiplicity free, s is a scalar multiple of $s_1 \cdots s_k$. Hence, $v(s) = v(s_1) + \dots + v(s_k)$. This finishes the proof. \square

Corollary 8.2. *The Okounkov body $\Delta(\mathcal{L}, N_L^+, v)$ is the convex hull of the points $(1, v(s_{\mathbf{m}}))$ with $1 \geq m_1 \geq \dots \geq m_r \geq 0$.*

Let $\mathcal{P} \subseteq \mathfrak{t}^*$ denote the weight lattice of \mathfrak{t} . Define the *moment semigroup*

$$S(\mathcal{L}, N_L^+) := \{(k, i\xi) \in \mathbb{N} \times \mathcal{P} \mid \xi \text{ is the highest weight of some } W_{(k\lambda, \mathbf{m})}\}.$$

We now construct a morphism of semigroups, Λ , from $S(\mathcal{L}, N_L^+, v)$ to $S(\mathcal{L}, N_L^+)$. As already observed in the proof of Proposition 8.1, two weight vectors $s_1, s_2 \in H^0(X, \mathcal{L}^k)$ with distinct weights cannot have the same value under v . Now, let $(k, a) \in S(\mathcal{L}, N_L^+, v)$. Since the decomposition of $H^0(X, \mathcal{L}^k)$ is multiplicity free there exists a unique (up to scalar multiples) weight vector $s \in H^0(X, \mathcal{L}^k)^{N_L^+}$ with $v(s) = a$. We therefore define

$$\Lambda : S(\mathcal{L}, N_L^+, v) \rightarrow \mathcal{P}$$

by putting $\Lambda((k, a)) = (k, i\xi)$, where ξ is the weight of s .

Remark 8.3. The morphism Λ was introduced by Okounkov in [Ok96] in a slightly different (but more general) setting.

Proposition 8.4. *The map $\Lambda : S(\mathcal{L}, N_L^+, v) \rightarrow S(\mathcal{L}, N_L^+)$ is an isomorphism of semigroups.*

Proof. We first prove that Λ is injective. For this, assume that for some $a \neq b$ we have $\Lambda((k, a)) = \Lambda((k, b))$. If $s, t \in H^0(X, \mathcal{L}^k)^{N_L^+}$ are highest weight vectors with $v(s) = a$ and $v(t) = b$, respectively, then s and t are linearly independent highest weight vectors of the same weight. This would, however, contradict the multiplicity-freeness of the decomposition of $H^0(X, \mathcal{L}^k)$ under L .

For the surjectivity, we note that, again by multiplicity-freeness, every highest weight ξ of some $W_{(k\lambda, \mathbf{m})}$ is the weight of a unique (up to scalar multiples) highest weight vector $s \in H^0(X, \mathcal{L}^k)^{N_L^+}$. Then $\Lambda((k, v(s))) = i\xi$. \square

Let $E(\mathcal{L}, N_L^+, v) \subseteq \mathbb{R} \times \mathbb{R}^n$ be the \mathbb{R} -linear subspace generated by $S(\mathcal{L}, N_L^+, v)$, and let $E(\mathcal{L}, N_L^+) \subseteq \mathbb{R} \times \mathfrak{t}^*$ be the \mathbb{R} -linear subspace generated by $S(\mathcal{L}, N_L^+)$. We extend Λ to a unique linear map $E(\mathcal{L}, N_L^+, v) \rightarrow E(\mathcal{L}, N_L^+)$, and we let Λ also denote this extension.

Let $C(\mathcal{L}, N_L^+) \subseteq E(\mathcal{L}, N_L^+)$ be the closed convex cone generated by $S(\mathcal{L}, N_L^+)$, and put

$$\Delta(\mathcal{L}, N_L^+) := C(\mathcal{L}, N_L^+) \cap (\{1\} \times \mathfrak{t}^*).$$

Clearly, we can identify $\Delta(\mathcal{L}, N_L^+)$ with Π_s by

$$\Delta(\mathcal{L}, N_L^+) = \{(1, x) \mid x \in \Pi_s\}.$$

Using this identification, Proposition 8.4 readily yields the following identification of the Okounkov body $\Delta(\mathcal{L}, N_L^+, v)$ with the moment polytope Π_s .

Theorem 8.5. *The map $\Lambda : E(\mathcal{L}, N_L^+, v) \rightarrow E(\mathcal{L}, N_L^+)$ restricts to a bijection*

$$\Delta(\mathcal{L}, N_L^+, v) \rightarrow \Pi_5.$$

Remark 8.6. The problem of constructing polyhedral Okounkov bodies has been addressed both in the setting of group actions, such in [Ok96], as well as in the case, developed in [LM09], when the semi-group is defined by the values of all sections. Few positive results in this direction are known, however. It is known to work for torus-equivariant line bundles over toric varieties (cf. [LM09]). As examples in the setting of homogeneous spaces under a reductive group we would like to mention [Ok98] and, more recently, [K11].

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BENJAMIN SCHWARZ, UNIVERSITÄT PADERBORN, FAKULTÄT FÜR ELEKTROTECHNIK, INFORMATIK UND MATHEMATIK, INSTITUT FÜR MATHEMATIK, WARBURGER STR. 100, 33098 PADERBORN

E-mail address: `bschwarz@math.upb.de`

HENRIK SEPPÄNEN, UNIVERSITÄT PADERBORN, FAKULTÄT FÜR ELEKTROTECHNIK, INFORMATIK UND MATHEMATIK, INSTITUT FÜR MATHEMATIK, WARBURGER STR. 100, 33098 PADERBORN

E-mail address: `henriksp@math.upb.de`